J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Supercongruences on some binomial sums involving Lucas sequences ☆

Guo-Shuai Mao, Hao Pan *

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

ARTICLE INFO

Article history:
Received 11 May 2016
Available online xxxx
Submitted by S. Cooper

Keywords:
Pell number
Central binomial coefficient
Congruence

ABSTRACT

In this paper, we confirm several conjectured congruences of Sun concerning the divisibility of binomial sums. For example, with help of a quadratic hypergeometric transformation, we prove that

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}^2\frac{P_k}{8^k}\equiv 0 \pmod{p^2}$$

for any prime $p\equiv 7 \pmod 8$, where P_k is the k-th Pell number. Further, we also propose three new congruences of the same type.

1. Introduction

In [10], with help of the Gross–Koblitz formula, Mortenson solved a conjecture of Rodriguez-Villegas [16] as follows:

$$\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{1}{16^k}\equiv\left(\frac{-1}{p}\right) \pmod{p^2}$$

for every odd prime p, where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. Subsequently, the similar congruences were widely studied. For the progress of this topic, the reader may refer to [11,12,14,6,17–19,8,20,7,4,21]. In [22], Sun proposed many conjectured congruences on the sums of binomial coefficients. Some of those conjectures are of the form

☆ This work is supported by the National Natural Science Foundation of China (Grant No. 11671197).

* Corresponding author.
E-mail addresses: mg1421007@smail.nju.edu.cn (G.-S. Mao), haopan79@zoho.com (H. Pan).

http://dx.doi.org/10.1016/j.jmaa.2016.10.055
0022-247X/© 2016 Elsevier Inc. All rights reserved.

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}^2 a_n \equiv 0 \pmod{p^2}.$$

For example, Sun conjectured that

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}^2 \frac{\chi_3(k)}{16^k} \equiv 0 \pmod{p^2}$$

for any prime $p \equiv 1 \pmod{12}$, where $\chi_3(k)$ equals to the Legendre symbol $\left(\frac{k}{3}\right)$.

The main purpose of this paper is to confirm the following conjectures of Sun.

Theorem 1.1. *Suppose that p is a prime.*

(i) *If $p \equiv 3 \pmod 4$, then*

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}^2 \frac{1}{(-8)^k} \equiv 0 \pmod{p^2}. \tag{1.1}$$

(ii) *If $p \equiv 1 \pmod{12}$, then*

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}^2 \frac{\chi_3(k)}{16^k} \equiv 0 \pmod{p^2}. \tag{1.2}$$

(iii) *If $p \equiv 7 \pmod 8$, then*

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}^2 \frac{P_k}{8^k} \equiv 0 \pmod{p^2}, \tag{1.3}$$

where the Pell number P_k is given by

$$P_0 = 0,\ P_1 = 1,\ P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geq 2.$$

(iv) *If $p \equiv 11 \pmod{12}$, then*

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{R_k}{(-4)^k}\binom{2k}{k}^2 \equiv 0 \pmod{p^2}, \tag{1.4}$$

where R_k is given by

$$R_0 = 2,\ R_1 = 4,\ R_n = 4R_{n-1} - R_{n-2} \text{ for } n \geq 2.$$

We mention that (1.1), (1.2), (1.3) and (1.4) respectively belong to Conjecture 5.5 of [18] and Conjectures A56, A57, A63 of [22].

The sequences $\{P_n\}$ and $\{R_n\}$ in Theorem 1.1 both belong to the second-order linear recurrence sequence. In general, define the Lucas sequences $\{U_n(a,b)\}$ and $\{V_n(a,b)\}$ by

$$U_0(a,b) = 0,\ U_1(a,b) = 1,\ U_n(a,b) = aU_{n-1}(a,b) - bU_{n-2}(a,b) \text{ for } n \geq 2,$$

and