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Number of synchronized and segregated interior spike solutions for nonlinear coupled elliptic systems

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A B S T R A C T

In this paper, we consider the following nonlinear coupled elliptic systems

$$\begin{cases} -\varepsilon^2\Delta u + u = \mu_1 u^3 + \beta uv^2 & \text{in } \Omega, \\ -\varepsilon^2\Delta v + v = \mu_2 v^3 + \beta u^2 v & \text{in } \Omega, \\ u > 0, v > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{A}_\varepsilon$}$$

where $\varepsilon > 0$, $\mu_1 > 0$, $\mu_2 > 0$, $\beta \in \mathbb{R}$, and Ω is a bounded domain with smooth boundary in $\mathbb{R}^3$. Due to Lyapunov–Schmidt reduction method, we proved that $(\mathcal{A}_\varepsilon)$ has at least $O(\frac{1}{\varepsilon^3|\ln\varepsilon|})$ synchronized and segregated vector solutions for ε small enough and some $\beta \in \mathbb{R}$. Moreover, for each $m \in (0, 3)$ there exist synchronized and segregated vector solutions for $(\mathcal{A}_\varepsilon)$ with energies in the order of ε^{3-m}. Our result extends the result of Lin, Ni and Wei [20], from the Lin–Ni–Takagi problem to the nonlinear elliptic systems.

1. Introduction and main results

In this paper, we study the following nonlinear elliptic systems

$$\begin{cases} -\varepsilon^2\Delta u + u = \mu_1 u^3 + \beta uv^2 & \text{in } \Omega, \\ -\varepsilon^2\Delta v + v = \mu_2 v^3 + \beta u^2 v & \text{in } \Omega, \\ u > 0, v > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{B}_\varepsilon$}$$

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where $\varepsilon > 0$, $\mu_1 > 0$, $\mu_2 > 0$, $\beta \in \mathbb{R}$, Ω is a bounded domain with smooth boundary in $\mathbb{R}^3$.

For $\Omega = \mathbb{R}^N, N \leq 3$ and $\varepsilon = 1$, $(\mathcal{B}_\varepsilon)$ leads to investigate the following problem

$$\begin{cases} -\Delta u + u = \mu_1 u^3 + \beta u v^2 & \text{in } \mathbb{R}^N, \\ -\Delta v + v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \\ u > 0, v > 0 & \text{in } \mathbb{R}^N, \\ u \to 0, v \to 0 & \text{as } x \to +\infty. \end{cases} \tag{1.1}$$

Problem $(\mathcal{B}_\varepsilon)$ arises in the Hartree–Fock theory for a double condensate, i.e. a binary mixture of Bose–Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ (see [9]). Physically, u and v are the corresponding condensate amplitudes. μ_j and β are the intraspecies and interspecies scattering lengths. The sign of scattering length β determines whether the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive or attractive. When $\beta > 0$, the interactions of states $|1\rangle$ and $|2\rangle$ are attractive, the components of a vector solution tend to go along with each other, leading to synchronization. In contrast, when $\beta < 0$, the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive, the components tend to segregate from each other, leading to phase separations.

Recently, B. Sirakov [24] discussed the whole $\beta \in \mathbb{R}$ and analyzed for which β the problem (1.1) assures a least energy solution and for which β the problem (1.1) has no least energy solution. In [31], Wei and Yao proved that for $0 < \beta < \min\{\mu_1, \mu_2\}$ small and $\beta > \max\{\mu_1, \mu_2\}$, the solution of (1.1) is unique and non-degenerate. Furthermore, Peng and Wang [23] proved there exists a decreasing sequence $\{\beta_k\} \subset (-\sqrt{\mu_1\mu_2}, 0)$ with $\beta_k \to -\sqrt{\mu_1\mu_2}$, the solution of (1.1) is non-degenerate for $\beta \in (-\sqrt{\mu_1\mu_2}, 0) \cup (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, +\infty)$ and $\beta \neq \beta_k$ for any k.

For the scalar case, there are many investigations to the following elliptic equations

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ u > 0, v > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and p is subcritical, i.e. $2 < p < \frac{2n}{n-2}$ for $n \geq 3$, and $2 < p < +\infty$ for $n = 1, 2$. It is well known that problem (1.2) has both interior spike solutions and boundary spike solutions. For single interior spike solutions, we refer to [7,28–30] and references therein. For multiple interior spikes solutions, we refer to [4,5,15] and references therein. For single boundary spike solutions, we refer to [8,19,21,22] and references therein. For multiple boundary spikes solutions, we refer to [6,13,16,17] and references therein. By Lyapunov–Schmidt reduction method, Gui and Wei [14] constructed a solution to (1.2) with both k interior spikes and l boundary spikes for any $k \geq 0$, $l \geq 0$, $k + l > 0$ and $\varepsilon > 0$ small enough.

We also want to mention the paper by Lin, Ni and Wei [20], where the authors showed that there exists ε_0 such that for each $0 < \varepsilon < \varepsilon_0$ and for each integer k bounded by $1 \leq k \leq \frac{C(\Omega,n)}{\varepsilon^n (\ln \varepsilon)^n}$, (1.2) has a solution with k-interior spikes. Recently, Ao, Wei and Zeng [1] promoted the upper bound of the number of interior spikes to $\frac{C(\Omega,n)}{\varepsilon^n}$ which is optimal due to the fact that each spike contributes to at least $O(\varepsilon^n)$ energy.

For the singularly perturbed nonlinear Schrödinger equations, we want to refer the readers to the work by Felmer, Martinez and Tanaka [10,11], where in 1-dimensional situation, they considered the following problem

$$\varepsilon^2 \Delta u - V(x) u + u^p = 0, u > 0, u \in H^1(\mathbb{R}). \tag{1.3}$$

They constructed a solution to (1.3) with $\frac{C}{\varepsilon}$ spikes for some $V(x)$.

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