J. Math. Anal. Appl. 448 (2017) 1120–1146

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On a singular elliptic system with quadratic growth in the gradient

Mohamed Benrhouma *

Mathematics Department, Faculty of Sciences of Monastir, University of Monastir, 5019 Monastir, Tunisia

ARTICLE INFO

Article history:
Received 25 August 2016
Available online 24 November 2016
Submitted by M. Musso

Keywords:
Singular elliptic systems
Quadratic growth in the gradient
Sub-supersolutions
Variational method
Nehari manifold
Concentration-compactness principle

ABSTRACT

In this paper, we deal with the following singular elliptic system:

$$\begin{cases} -\Delta u + \alpha \frac{|\nabla u|^2}{u} = \frac{p}{p+q} a(x)|v|^q|u|^{p-2}u + f, & x \in \mathbb{R}^N, \\ -\Delta v + \beta \frac{|\nabla v|^2}{v} = \frac{q}{p+q} a(x)|u|^p|v|^{q-2}v + g, & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, α, $\beta > \frac{N+2}{4}$, p, $q > 1$ and $p+q \leq \frac{N+2}{N-2}$. We show through the sub- and supersolutions method, the existence of a nonnegative solution for an approximated system. The limit of the approximated solution is a positive solution. In the case, $\alpha = \beta = 0$, $p = q$ and $f = g$, we prove the uniqueness of a solution. Among others, we prove some existence and uniqueness results for some auxiliary problems by using the comparison principle, a minimization method and with the help of Nehari manifold. The proofs rely on the concentration-compactness principle.

1. Introduction

In this paper, we are concerned with the existence of solutions of the following system:

$$\begin{cases} -\Delta u + \alpha \frac{|\nabla u|^2}{u} = \frac{p}{p+q} a(x)|v|^q|u|^{p-2}u + f, & x \in \mathbb{R}^N, \\ -\Delta v + \beta \frac{|\nabla v|^2}{v} = \frac{q}{p+q} a(x)|u|^p|v|^{q-2}v + g, & x \in \mathbb{R}^N. \end{cases} \tag{1.1}$$

Our main hypotheses are the following:

* Fax: +966 137234659.
E-mail address: mbenrhouma@uod.edu.sa.

http://dx.doi.org/10.1016/j.jmaa.2016.11.038

(H_1) $N > 2$, $p > 1$, $q > 1$, $p + q \leq \dfrac{N+2}{N-2}$, α, $\beta > \dfrac{N+2}{4}$.

(H_2) $a(x) \geq 0$ a.e. in $\mathbb{R}^N$ and $a \in L^r(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$ where

$$r = \frac{2N}{2N-(p+q)(N-2)} \quad \text{and} \quad s = \frac{2N}{N+2-(p+q)(N-2)}.$$

(H_3) f, $g > 0$ a.e. in $\mathbb{R}^N$ and

$$f,\ g \in L^{\frac{2N}{N+2}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

Nonlinear or singular elliptic equations (and then systems) have been intensively studied during last decades. The study of this type of problem is motivated by its various applications such as fluid mechanics pseudo-plastics flow, chemical heterogeneous catalysts, non-Newtonian fluids, biological pattern formation. Several papers dealing with semilinear elliptic systems in the whole space $\mathbb{R}^N$. See, for example, [5,12,14,16,18,20,21,28,30,32,33,40]. Benrhouma [5] studied the existence of solutions of the following system:

$$\begin{cases} -\Delta u + u = \frac{\alpha}{\alpha+\beta} a(x)|v|^\beta |u|^{\alpha-2}u & \text{in } \mathbb{R}^N, \\ -\Delta v + v = \frac{\beta}{\alpha+\beta} a(x)|u|^\alpha |v|^{\beta-2}v & \text{in } \mathbb{R}^N. \end{cases} \tag{1.2}$$

With the help of the Nehari manifold and the linking theorem (appropriately modified), the author proved the existence of at least two solutions of (1.2). The supersolution and subsolution method is used by Kawano and Kusano [30] to establish the existence of infinitely positive entire solutions of the semilinear system:

$$\begin{cases} \Delta u + f(x,u,v) = 0, & x \in \mathbb{R}^N, \\ \Delta v + g(x,u,v) = 0, & x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

Semilinear elliptic systems with quadratic growth in the gradient have been considered in [6,24,36,43]. Miao and Yang [36] established the existence of infinitely positive bounded entire solutions of the following system:

$$\begin{cases} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + f(x,u,v) = 0, & x \in \mathbb{R}^N, \\ \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) + g(x,u,v) = 0, & x \in \mathbb{R}^N. \end{cases} \tag{1.4}$$

Zhang and Liu [43] studied the existence and the nonexistence of entire positive solutions for the system:

$$\begin{cases} \Delta u + |\nabla u| = p(|x|)f(u,v), & x \in \mathbb{R}^N, \\ \Delta v + |\nabla v| = q(|x|)g(u,v), & x \in \mathbb{R}^N. \end{cases} \tag{1.5}$$

It seems to us that few results are known on singular elliptic systems. We can for example quote [7,17,22,25,29,35,37]. Benrhouma [7] proved the existence and uniqueness of solutions for the following singular system:

$$\begin{cases} -\Delta u + f(x)u^2 = g(x)u + \frac{1-p}{2-p-q} a(x)|u|^{-p}|v|^{1-q}, & x \in \mathbb{R}^N, \\ -\Delta v + f(x)v^2 = g(x)v + \frac{1-q}{2-p-q} a(x)|u|^{1-p}|v|^{-q}, & x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$

where $0 < p < 1$ and $0 < q < 1$. Colin and Frigon [17] established the existence of a nontrivial solution to systems of singular Poisson equations in unbounded domains Ω: