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The multinomial convolution sums of certain divisor functions [☆]

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A B S T R A C T

In this article we first introduce some results about the binomial, trinomial, and quadrinomial convolution sums of certain divisor functions and then we provide an identity for the multinomial convolution sums of the divisor function $\sigma_r^\sharp(n) := \sum_{\substack{d|n \\ n/d\equiv 1(4)}} d^r - (-1)^r \sum_{\substack{d|n \\ n/d\equiv -1(4)}} d^r$.

1. Introduction

We define $\sigma_r(n) = \sum_{d|n} d^r$ $(r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ and $\sigma(n) = \sigma_1(n)$. The well-known identity

$$\sum_{m=1}^{n-1} \sigma(m)\sigma(n-m) = \frac{5}{12}\sigma_3(n) + \left(\frac{1}{12} - \frac{1}{2}n\right)\sigma(n) \tag{1}$$

for the binomial convolution sum of the divisor function $\sigma(n)$ first appeared in a letter from Besge to Liouville in 1862 (see [1]). An identity for the binomial convolution sum of the divisor function $\sigma_r(n)$ was obtained by Liouville [6].

Proposition 1.1 *([6]). For each $k, n \in \mathbb{N}$, we have*

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m) = \frac{2k+3}{4k+2}\sigma_{2k+1}(n)$$

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$$+\left(\frac{k}{6}-n\right)\sigma_{2k-1}(n)+\frac{1}{2k+1}\sum_{j=2}^{k}\frac{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(n), \tag{2}$$

where B_j ($j \in \mathbb{N}_0$) is the j-th Bernoulli number.

In 2013 Kim and Bayad [3] gave an identity for the binomial convolution sum of the divisor function $\sigma_r^*(n) := \sum_{\substack{d|n \\ 2\nmid n/d}} d^r$ ($r \in \mathbb{N}_0$).

Proposition 1.2 *([3]). For each $k, n \in \mathbb{N}$, we have*

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}^*(m)\sigma_{2s+1}^*(n-m)=\frac{1}{2}\left(\sigma_{2k+1}^*(n)-n\sigma_{2k-1}^*(n)\right). \tag{3}$$

Recently, Kim and Park [5] proved that the trinomial and quadrinomial convolution sums of the same divisor function are as follows:

Proposition 1.3 *([5]). Let $n \geq 4$ be an even integer with $k \in \mathbb{N}$. Then*

$$(i)\quad \sum_{\substack{a+b+c=2k+1 \\ a,b,c \text{ odd}}} a\binom{2k+1}{a,b,c}\sum_{\substack{m_1+m_2+m_3=n \\ m_3 \text{ even}}}(-1)^{m_1+1}\sigma_a^*(m_1)\sigma_b^*(m_2)\sigma_c^*(m_3)$$
$$=\frac{(2k-1)n}{32}(\sigma_{2k+1}^*(n)-2n\sigma_{2k-1}^*(n)),$$
$$(ii)\quad \sum_{\substack{a+b+c+d=2k \\ a,b,c,d \text{ odd}}}\frac{a+b}{c+d+1}\binom{2k}{a,b,c,d}\sum_{\substack{m_1+m_2+m_3+m_4=n \\ m_3,m_4 \text{ odd}}}(-1)^{m_1+1}\sigma_a^*(m_1)\sigma_b^*(m_2)\sigma_c^*(m_3)\sigma_d^*(m_4)$$
$$=\frac{1}{64}\left(n\sigma_{2k+1}^*(n)-2n^2\sigma_{2k-1}^*(n)-64\sum_{m<n/2}(n/2-m)\sigma_1^*(m)\sigma_{2k-1}^*(n-2m)\right).$$

Now we define the following divisor functions. For $N \geq 3, 1 \leq i \leq N-1, r \geq 0$ and $n \in \mathbb{N}$

$$\sigma_r^\sharp(n;i,N)=\sum_{\substack{d|n \\ \frac{n}{d}\equiv i(N)}}d^r-(-1)^r\sum_{\substack{d|n \\ \frac{n}{d}\equiv -i(N)}}d^r$$
$$\sigma_r^\flat(n;i,N)=\sum_{\substack{d|n \\ d\equiv i(N)}}d^r-(-1)^r\sum_{\substack{d|n \\ d\equiv -i(N)}}d^r$$

and set $\sigma^\sharp(n;i,N)=\sigma_1^\sharp(n;i,N)$ and $\sigma^\flat(n;i,N)=\sigma_1^\flat(n;i,N)$.

The purpose of this article is to give some identities for these divisor functions. First we obtain results for the binomial convolution sums of $\sigma_r^\sharp(n;i,N)$ and $\sigma_r^\flat(n;i,N)$.

Theorem 1.4. *Suppose $N \geq 3$ and $1 \leq i \leq N-1$. For all $k, n \in \mathbb{N}$ we have*

$$\sum_{s=0}^{2k}\binom{2k}{s}\sum_{m=1}^{n-1}\sigma_s^\sharp(n-m;i,N)\sigma_{2k-s}^\sharp(m;i,N)$$
$$=\sigma_{2k+1}^\sharp(n;i,N)-\left(1-\frac{2i}{N}\right)\sigma_{2k}^\sharp(n;i,N)-\frac{2n}{N}\sigma_{2k-1}^\sharp(n;i,N) \tag{4}$$

and