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Gevrey regularity of mild solutions to the parabolic–elliptic system of drift-diffusion type in critical Besov spaces ☆

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ABSTRACT

This paper is devoted to studying analyticity of mild solutions to the parabolic–elliptic system of drift-diffusion type with small initial data in critical Besov spaces. More precisely, by using multilinear singular integrals and Fourier localization argument, we show that global-in-time mild solutions are Gevrey regular, i.e., $(e^{\sqrt{t}\Lambda_1}v, e^{\sqrt{t}\Lambda_1}w) \in \widetilde{L}^\infty_t \dot{B}^{-2+\frac{n}{p}}_{p,r}(\mathbb{R}^n) \cap \widetilde{L}^1_t \dot{B}^{\frac{n}{p}}_{p,r}(\mathbb{R}^n)$ for all $1 < p < 2n$ and $1 \le r \le \infty$, where Λ_1 is the Fourier multiplier whose symbol is given by $|\xi|_1 = \sum_{i=1}^n |\xi_i|$. As a corollary, we obtain decay estimates of mild solutions in critical Besov spaces without using cumbersome recursive estimation of higher-order derivatives.

1. Introduction

In this paper, we study analyticity of mild solutions for the initial value problem of the following drift-diffusion system arising from the theory of electrolytes (cf. [14,37]):

$$\begin{cases} \partial_t v = \Delta v - \nabla\cdot(v\nabla\phi) & \text{in } \mathbb{R}^n\times(0,\infty),\\ \partial_t w = \Delta w + \nabla\cdot(w\nabla\phi) & \text{in } \mathbb{R}^n\times(0,\infty),\\ \Delta\phi = v - w & \text{in } \mathbb{R}^n\times(0,\infty),\\ v(x,0)=v_0(x),\quad w(x,0)=w_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where the unknown functions $v = v(x,t)$ and $w = w(x,t)$ denote densities of the electron and the hole in electrolytes, respectively, $\phi = \phi(x,t)$ denotes the electric potential, $v_0(x)$ and $w_0(x)$ are initial datum. Throughout this paper, we assume that $n \ge 2$.

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We first recall the definition of a mild solution (v, w) to the system (1.1). Notice the fact that the electric potential ϕ is determined by the Poisson equation, the third equation of (1.1), gives rise to the coefficient $\nabla\phi$ in the first two equations of (1.1), so when ϕ is represented as the volume potential of v, w:

$$\phi(x,t) = (-\Delta)^{-1}(w-v)(x,t) = \begin{cases} \frac{1}{n(n-2)\Omega_n}\int_{\mathbb{R}^n} \frac{w(y,t)-v(y,t)}{|x-y|^{n-2}}dy, & n \geq 3, \\ -\frac{1}{2\pi}\int_{\mathbb{R}^2}(w(y,t)-v(y,t))\log|x-y|dy, & n = 2, \end{cases}$$

where Ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$, the system (1.1) can be rewritten as the differential-integral Fokker–Planck system via the well known Duhamel principle:

$$\begin{cases} v = e^{t\Delta}v_0 - \int_0^t e^{(t-\tau)\Delta}\nabla\cdot[v\nabla(-\Delta)^{-1}(w-v)]d\tau, \\ w = e^{t\Delta}w_0 + \int_0^t e^{(t-\tau)\Delta}\nabla\cdot[w\nabla(-\Delta)^{-1}(w-v)]d\tau, \end{cases} \tag{1.2}$$

where $e^{t\Delta} = \mathcal{F}^{-1}(e^{-t|\xi|^2}\mathcal{F})$ is the heat flow operator, $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ is the inverse Fourier transform. Any solution satisfying the integral system (1.2) is called a *mild solution* of the system (1.1).

Notice that the system (1.1) is scaling invariant in the following sense: if (v, w) solves (1.1) with initial data (v_0, w_0) (ϕ can be determined by (v, w)), so does (v_λ, w_λ) with initial data $(v_{0\lambda}, w_{0\lambda})$ (ϕ_λ can be determined by (v_λ, w_λ)), where

$$(v_\lambda(x,t), w_\lambda(x,t)) := (\lambda^2 v(\lambda x, \lambda^2 t), \lambda^2 w(\lambda x, \lambda^2 t)), \quad (v_{0\lambda}(x), w_{0\lambda}(x)) := (\lambda^2 v_0(\lambda x), \lambda^2 w_0(\lambda x)).$$

A function space $E \subset \mathcal{S}'(\mathbb{R}^n)$ is said to be a *critical space* for initial data of the system (1.1) if the norms of $(v_{0\lambda}(x), w_{0\lambda}(x))$ in E are equivalent for all $\lambda > 0$, i.e.,

$$\|(v_{0\lambda}(x), w_{0\lambda}(x))\|_E \approx \|(v_0(x), w_0(x))\|_E.$$

Under these scalings, it is clear that $L^{\frac{n}{2}}(\mathbb{R}^n)$, $\dot{H}^{-2+\frac{n}{2}}(\mathbb{R}^n)$ and $\dot{B}_{p,r}^{-2+\frac{n}{p}}(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ and $1 \leq r \leq \infty$ are critical spaces for initial data of the system (1.1). In particular, the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ and Besov space $\dot{B}_{1,2}^0(\mathbb{R}^2)$ are critical spaces for initial data of two dimensional system (1.1).

The system (1.1) has been extensively studied in the past three decades, see [3–5,18,19,31–33,36,39] and references therein. In two dimensional case, Ogawa and Shimizu in [34,35] established the local-in-time well-posedness for large initial data and the global-in-time well-posedness for small initial data in critical Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ and critical Besov space $\dot{B}_{1,2}^0(\mathbb{R}^2)$, respectively. Recently, Deng and Li [15] established a dichotomy for well-posedness and ill-posedness issues of the system (1.1), more precisely, they proved that the system (1.1) is well-posed in $\dot{B}_{4,2}^{-\frac{3}{2}}(\mathbb{R}^2)$, and ill-posed in $\dot{B}_{4,r}^{-\frac{3}{2}}(\mathbb{R}^2)$ for $2 < r \leq \infty$. In general dimensional case, Karch [25] proved existence of global-in-time mild solution of the system (1.1) with small initial data in critical Besov space $\dot{B}_{p,\infty}^{-2+\frac{n}{p}}(\mathbb{R}^n)$ with $\frac{n}{2} \leq p < n$. We mention here that similar results for initial data in subcritical and critical Lebesgue and Sobolev spaces were established only recently, see the work of Kurokiba and Ogawa [27]. Very recently, the author of this paper, Liu and Cui [42] proved that small data global existence and large data local existence of mild solutions of the system (1.1) in critical Besov space $\dot{B}_{p,r}^{-2+\frac{n}{p}}(\mathbb{R}^n)$ of indices $1 < p < 2n$ and $1 \leq r \leq \infty$.

Our aim in this paper is to show analyticity of mild solutions to the system (1.1) by establishing Gevrey regularity. The use of Gevrey regularity was pioneered by Foias and Temam [16,17] for estimating space analyticity radius of the Navier–Stokes equations, since then, it was subsequently generalized by many authors to different equations and various functional spaces, see [1,8–10,20,28,38]. In this paper, we shall show that mild solutions obtained in [42] are actually Gevrey regular, that is to say, $(e^{\sqrt{t}\Lambda_1}v, e^{\sqrt{t}\Lambda_1}w) \in \mathcal{X}_{p,r}$