Doctopic: Complex Analysis ARTICLE IN PRESS YJMAA:20910

J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Disjoint universality for families of Taylor-type operators

V. Vlachou

Department of Mathematics, University of Patras, 26500 Patras, Greece

ARTICLE INFO

Article history:
Received 19 May 2016
Available online xxxx
Submitted by R.M. Aron

Keywords:
Universal Taylor series
Multiple universality
Disjoint universality

ABSTRACT

We give a necessary and sufficient condition so that we have disjoint universality for sequences of operators that map a holomorphic function to a partial sum of its Taylor expansion. This problem is connected with doubly universal Taylor series and this is an effort to generalize the concept to multiply universal Taylor series.

1. Introduction

In the last 30 years, several authors have worked on the notion of universality and important advances in the research of this topic have been made. Roughly speaking, we say that a family of operators is universal if it has a dense orbit. More recent papers have introduced and studied a new notion: the notion of disjoint universality (see [4] and [5]; see also [3] and [13]). This notion involves several families of operators. We say that a collection of families of operators is disjoint universal if there exists a common vector with dense orbit for all families, such that the approximation of any fixed vector is simultaneously performed with a common subsequence. Our goal is to study disjoint universality for families of Taylor-type Operators i.e. to work with the notion of universal and doubly universal Taylor series. To be more specific, let us fix a simply connected domain $\Omega \subset \mathbb{C}$. We denote by $H(\Omega)$ the space of functions holomorphic in Ω, endowed with the topology of uniform convergence on compacta. Moreover, for $n \in \mathbb{N}$, $\zeta_0 \in \Omega$ and $f \in H(\Omega)$ we denote by $S_n(f, \zeta_0)$ the nth partial sum of the Taylor expansion of f around ζ_0.

Finally we also use the notations

- $\mathcal{A}(K) = \{g \in H(K^o) : \ g \text{ is continuous on } K\}$, $K \subset \mathbb{C}$ compact
- $||g||_K = \sup\limits_{z \in K} |g(z)|$, $g \in \mathcal{A}(K)$

E-mail address: vvlachou@math.upatras.gr.

http://dx.doi.org/10.1016/j.jmaa.2016.11.057
0022-247X/© 2016 Published by Elsevier Inc.

- $\mathcal{M} = \{K \subset \mathbb{C} : K$ is a compact set with K^c connected set$\}$
- $\mathcal{M}_{\Omega^c} = \{K \in \mathcal{M} : K \subset \Omega^c\}$.

We are now ready to give the definition of Universal Taylor Series as given in [21] (see also [20]):

Definition 1.1. A holomorphic function $f : \Omega \to \mathbb{C}$ belongs to the class $U(\Omega, \zeta_0)$, for a fixed point $\zeta_0 \in \Omega$, if the set $\{S_n(f, \zeta_0) : \ n \in \mathbb{N}\}$ is dense in $\mathcal{A}(K)$ (endowed with the norm $|| \cdot ||_K$) for every $K \in \mathcal{M}_{\Omega^c}$.

In [21] V. Nestoridis proved that the class $U(\Omega, \zeta_0)$ is a G_δ and dense subset of $H(\Omega)$ for every $\zeta_0 \in \Omega$. We proceed by giving the definition of doubly universal Taylor Series as given in [6]:

Definition 1.2. A holomorphic function $f : \Omega \to \mathbb{C}$ belongs to the class $U(\Omega, (\lambda_n)_n, \zeta_0)$, for a fixed point $\zeta_0 \in \Omega$ and for a fixed sequence of positive integers $(\lambda_n)_n$, if the set of pairs $\{(S_n(f, \zeta_0), S_{\lambda_n}(f, \zeta_0)) : \ n \in \mathbb{N}\}$ is dense in $\mathcal{A}(K_1) \times \mathcal{A}(K_2)$ for every $K_1, K_2 \in \mathcal{M}_{\Omega^c}$.

Actually the above definition is due to G. Costakis and N. Tsirivas (see [7]) who were the first to consider pairs of partial sums motivated by disjoint universality. Their result gave a necessary and sufficient condition on $(\lambda_n)_n$ so that the class $U(\Omega, (\lambda_n)_n, \zeta_0)$ is non-empty for $\Omega = \mathbb{D}$ (the unit disk), $\zeta_0 = 0$ and $K_1 = K_2$. They presented brilliant ideas in both directions of proof. These ideas were refined and improved in [6], where the same result was proved for the general class of Definition 1.2.

Double universality as mentioned above is actually disjoint universality for two sequences of operators. The motivation of the present article arises from the natural question: what about more than two sequences of operators? To be more specific, let us give the definition of the class of functions we are interested in:

Definition 1.3. A holomorphic function $f \in H(\Omega)$ belongs to the class $U_{mult} \equiv U_{mult}(\Omega, \ [(\lambda_n^{(\sigma)})_n, \zeta_\sigma \ ]_{\sigma=1}^{\sigma_0})$, for a fixed finite collection of sequences of positive integers $(\lambda_n^{(\sigma)})_n$, $\sigma = 1, 2, \ldots \sigma_0$ and a corresponding fixed finite collection of points $\zeta_1, \zeta_2, \ldots, \zeta_{\sigma_0} \in \Omega$, if the set

$$\{(S_{\lambda_n^{(1)}}(f, \zeta_1), S_{\lambda_n^{(2)}}(f, \zeta_2), \ldots, S_{\lambda_n^{(\sigma_0)}}(f, \zeta_{\sigma_0})) : n \in \mathbb{N}\}$$

is dense in $\mathcal{A}(K_1) \times \mathcal{A}(K_2) \times \ldots \times \mathcal{A}(K_{\sigma_0})$, for every choice of sets $K_1, K_2, \ldots, K_{\sigma_0} \in \mathcal{M}_{\Omega^c}$.

In the main section of the present article (section 2), we give a necessary and sufficient condition on the choice of the sequences $(\lambda_n^{(\sigma)})_n$, $\sigma = 1, 2, \ldots \sigma_0$ so that the class U_{mult} is non-empty. We use many of the arguments presented in [7] and [6]. Those arguments were not enough though to deal with the case $\sigma_0 > 2$ or with the case of considering different centers of expansion $\zeta_1, \zeta_2, \ldots, \zeta_{\sigma_0}$. Our tools include concepts and theorems from potential theory for which we would like to refer to [23]. Lately, several authors have used potential theory in problems concerning universality (see [6–11,16,18,19,25]). We would also like to mention that this problem shares many similarities with the notion of joint universality which has been introduced and studied in analytic number theory (see for example [1,14,24,22]).

Finally, in the last section, we work with a special (finite) choice of sequences of positive integers and using Ostrowski-gaps we prove that the class U_{mult} is independent of the choice of centers of expansion. We use methods and ideas presented in [12] and [18] (see also [15] and [17]).

2. The general case

At the beginning of this section, we will provide a specific approximation theorem which will be useful to our goal. This theorem constitutes a modification of the well known Bernstein–Walsh Theorem (see p. 170 in [23]) which provides a tool to get a polynomial of at most a certain degree near a given function.