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# A note on the admissibility of modular function spaces

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ABSTRACT

In this paper we prove the admissibility of modular function spaces $E_\rho$ considered and defined by Kozłowski in [17]. As an application we get that any compact and continuous mapping $T : E_\rho \to E_\rho$ has a fixed point. Moreover, we prove that the same holds true for any retract of $E_\rho$.



## 1. Introduction

The notion of admissibility, introduced by Klee in [14], allows one to approximate the identity on compact sets by finite-dimensional mappings. Locally convex spaces are admissible (see [24]), and a large literature is devoted to prove that particular classes of non-locally convex function spaces are admissible, among others we mention [7,21,25–27]. Moreover, in [2] the admissibility of spaces of functions determined by finitely additive set functions has been proved. It is important to notice that not all non-locally convex spaces are admissible, in [3] Cauty provides an example of a metric linear space in which the admissibility fails. Here we prove the admissibility of modular function spaces in the framework defined by Kozłowski in [17] (see also [15,16]). Modular function spaces are a natural generalization of both function and sequence variants of Orlicz, Musielak–Orlicz, Lorentz, Orlicz–Lorentz, Calderón–Lozanovskii spaces and many others. Our interest in the admissibility of modular function spaces lies in the possibility of applying the result to the fixed point theory. The fixed point theory in modular function spaces was initiated by Khamsi, Kozłowski and Reich [13], and it is a topic of interest in the theory of nonlinear operators, see e.g. [1,4,5,8–11,18,20,22] and references therein. For more information about the current state of the theory the reader is referred to [12]. One of the advantages of the theory, as observed for example in [5], is that even in absence of a metric,

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many problems in metric fixed point theory can be reformulated in modular spaces. We recall the definition of admissibility.

**Definition 1.** [14] Let $E$ be a Haudorff topological vector space. A subset $Z$ of $E$ is said to be *admissible* if for every compact subset $K$ of $Z$ and for every neighborhood $V$ of zero in $E$ there exists a continuous mapping $H : K \to Z$ such that $\dim(\text{span}[H(K)]) < \infty$ and $f - Hf \in V$ for every $f \in K$. If $Z = E$ we say that the space $E$ is admissible.

## 2. Preliminaries

We start by introducing modular function spaces, following [17]. Let $X$ be a nonempty set and $\mathcal{P}$ a nontrivial $\delta$-ring of subsets of $X$, i.e. a ring closed under countable intersections. Let $\Sigma$ be the smallest $\sigma$-algebra of subsets of $X$ such that $\mathcal{P}$ is contained in $\Sigma$. Let us assume that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$, and

$$X = \bigcup_{n=1}^{\infty} X_n, \tag{1}$$

where $X_n \subset X_{n+1}$ and $X_n \in \mathcal{P}$ for any $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers. Let $(W, \|\cdot\|)$ be a Banach space. By a $\mathcal{P}$-simple function on $X$ with values in $W$ we mean a function of the form

$$g = \sum_{i=1}^{n} w_i \chi_{E_i},$$

where $\omega_i \in W$, $E_i \in \mathcal{P}$, $E_i \cap E_j = \emptyset$ for $i \neq j$, and by $\mathcal{E}$ we denote the linear space of all $\mathcal{P}$-simple functions. A function $f : X \to W$ is called *measurable* if there exists a sequence of $\mathcal{P}$-simple functions $\{s_n\}$ such that $s_n(x) \to f(x)$ for any $x \in X$. By $M(X, W)$ we denote the set of all measurable functions.

**Definition 2.** A functional $\rho : \mathcal{E} \times \Sigma \to [0, +\infty]$ is called a *function modular* if it satisfies the following properties:

- (P1) $\rho(0, E) = 0$ for every $E \in \Sigma$;
- (P2) $\rho(f, E) \le \rho(g, E)$ whenever $\|f(x)\| \le \|g(x)\|$ for all $x \in E$ and any $f, g \in \mathcal{E}$ $(E \in \Sigma)$;
- (P3) $\rho(f, \cdot) : \Sigma \to [0, +\infty]$ is a $\sigma$-subadditive measure for every $f \in \mathcal{E}$;
- (P4) $\rho(\alpha, A) \to 0$ as $\alpha$ decreases to 0 for every $A \in \mathcal{P}$, where for $\alpha > 0$

$$\rho(\alpha, A) = \sup\{\rho(r\chi_A, A) : r \in W, \|r\| \le \alpha\};$$

- (P5) there is $\alpha_0 \ge 0$ such that $\sup_{\beta>0} \rho(\beta, A) = 0$ whenever $\sup_{\alpha>\alpha_0} \rho(\alpha, A) = 0$;
- (P6) $\rho(\alpha, \cdot)$ is order continuous on $\mathcal{P}$ for every $\alpha > 0$, that is $\rho(\alpha, A_n) \to 0$ for any sequence $\{A_n\} \subset \mathcal{P}$ decreasing to $\emptyset$.

Then for $f \in M(X, W)$ we set

$$\rho(f, E) = \sup\{\rho(g, E) : g \in \mathcal{E}, \|g(x)\| \le \|f(x)\| \text{ for all } x \in E\}.$$

**Definition 3.** A set $E \in \Sigma$ is said to be *ρ-null* if $\rho(\alpha, E) = 0$ for every $\alpha > 0$, and a property is said to be hold *ρ-almost everywhere* (briefly *ρ-a.e.*) if the set where it fails to hold is $\rho$-null.