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Spectral triples for noncommutative solenoidal spaces from self-coverings

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ABSTRACT

Examples of noncommutative self-coverings are described, and spectral triples on the base space are extended to spectral triples on the inductive family of coverings, in such a way that the covering projections are locally isometric. Such triples are shown to converge, in a suitable sense, to a semifinite spectral triple on the direct limit of the tower of coverings, which we call noncommutative solenoidal space. Some of the self-coverings described here are given by the inclusion of the fixed point algebra in a C*-algebra acted upon by a finite abelian group. In all the examples treated here, the noncommutative solenoidal spaces have the same metric dimension and volume as on the base space, but are not quantum compact metric spaces, namely the pseudo-metric induced by the spectral triple does not produce the weak* topology on the state space.

Contents

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0. Introduction

Given a noncommutative self-covering consisting of a C*-algebra with a unital injective endomorphism $(\mathcal{A}, \alpha)$, we study the possibility of extending a spectral triple on $\mathcal{A}$ to a spectral triple on the inductive limit C^*-algebra, where, as in [19], the inductive family associated with the endomorphism α is

$$\mathcal{A}_0 \xrightarrow{\alpha} \mathcal{A}_1 \xrightarrow{\alpha} \mathcal{A}_2 \xrightarrow{\alpha} \mathcal{A}_3 \dots, \tag{0.1}$$

all the $\mathcal{A}_n$ being isomorphic to $\mathcal{A}$. The algebra $\mathcal{A}_n$ may be considered as the n-th covering of the algebra $\mathcal{A}_0$ w.r.t. the endomorphism α. As a remarkable byproduct, all the spectral triples we construct on the inductive limit C^*-algebra are semifinite spectral triples.

Let us recall that the first notion of type II noncommutative geometry appeared in [18], where semifinite Fredholm modules were introduced, a notion then generalized in [11], see also [12], with that of semifinite unbounded Fredholm module. The latter is essentially the same definition as that of von Neumann spectral triples of [7], where some previous constructions [4,15,36,27] were reinterpreted as examples of such concept. In the same period, Ref. [29] considered semifinite spectral triples for graph algebras and posed the problem of exhibiting more examples of the kind, which was done in [30] using k-graph algebras and in [1] inspired by quantum gravity. Further examples have been considered in [38,23,10].

In the cases we analyze, it is possible to construct natural spectral triples on the C*-algebras $\mathcal{A}_n$ of the inductive family, which converge, in a suitable sense, to a triple on the inductive limit, and the latter triple is indeed semifinite.

The leading idea is that of producing geometries on each of the noncommutative coverings $\mathcal{A}_n$ which are locally isomorphic to the geometry on the original noncommutative space $\mathcal{A}$. This means in particular that the covering projections should be local isometries or, in algebraic terms, that the noncommutative metrics given by the Lip-norms associated with the Dirac operators via $L_n(a) = \|[D_n, a]\|$ (cf. [16,34]) should be compatible with the inductive maps, i.e. $L_{n+1}(\alpha(a)) = L_n(a)$, $a \in \mathcal{A}_n$. In one case, this property will be weakened to the existence of a finite limit for the sequences $L_{n+p}(\alpha^p(a))$, $a \in \mathcal{A}_n$.

The above request produces two related effects. On the one hand, the noncommutative coverings are metrically larger and larger, so that their radii diverge to infinity, and *the inductive limit is topologically compact* (the C*-algebra has a unit) *but not totally bounded* (the metric on the state space does not induce the weak*-topology). On the other hand, the spectrum of the Dirac operator becomes more and more dense in the real line, so that the resolvent of the limiting Dirac operator cannot be compact, being indeed τ-compact w.r.t. a suitable trace, and thus producing a *semifinite spectral triple on the inductive limit.*

Pursuing this idea means also that we see the elements of the inductive family in a more geometric way, namely as distinct (though isomorphic) algebras of "functions" on noncommutative coverings, and the inductive maps as embeddings of a sub-algebra into an algebra of "less periodic" functions. In this sense the