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Exponents of operator self-similar random fields ☆

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ABSTRACT

If $X(c^E t)$ and $c^H X(t)$ have the same finite-dimensional distributions for some pair of linear operators E and H, we say that the random vector field $X(t)$ is operator self-similar. The exponents E and H are not unique in general, due to symmetry. This paper characterizes the possible set of range exponents H for a given domain exponent, and conversely, the set of domain exponents E for a given range exponent.

1. Introduction

A random vector is called *full* if its distribution is not supported on a lower dimensional hyperplane. A random field $X = \{X(t)\}_{t\in\mathbb{R}^m}$ with values in $\mathbb{R}^n$ is called *proper* if $X(t)$ is full for all $t \neq 0$. A linear operator P on $\mathbb{R}^m$ is called a projection if $P^2 = P$. Any nontrivial projection $P \neq I$ maps $\mathbb{R}^m$ onto a lower dimensional subspace. We say that a random vector field X is *degenerate* if there exists a nontrivial projection P such that $X(t) = X(Pt)$ for all $t \in \mathbb{R}^m$. We say that X is *stochastically continuous* if $X(t_n) \to X(t)$ in probability whenever $t_n \to t$. A proper, nondegenerate, and stochastically continuous random vector field X is called *operator self-similar* (o.s.s., or (E, H)-o.s.s.) if

$$\{X(c^E t)\}_{t\in\mathbb{R}^m} \simeq \{c^H X(t)\}_{t\in\mathbb{R}^m} \quad \text{for all } c > 0. \tag{1.1}$$

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In (1.1), $\simeq$ indicates equality of finite-dimensional distributions, $E \in M(m, \mathbb{R})$ and $H \in M(n, \mathbb{R})$, where $M(p, \mathbb{R})$ represents the space of real-valued $p \times p$ matrices, and $c^M = \exp(M(\log c)) = \sum_{k=0}^{\infty}(M \log c)^k/k!$ for a square matrix M. We will assume throughout this paper that the eigenvalues of E and H have (strictly) positive real parts. This ensures that $c^E t$ and $c^H x$ tend to zero as $c \to 0$, and tend to infinity in norm as $c \to \infty$ for any $t, x \neq 0$, see Theorem 2.2.4 in Meerschaert and Scheffler [23]. Then it follows from stochastic continuity that $X(0) = 0$ a.s. At the end of Section 2, we will discuss what happens if some eigenvalues of H have zero real part.

Operator self-similar random (vector) fields are useful to model long-range dependent, spatial and spatio-temporal anisotropic data in hydrology, radiology, image processing, painting and texture analysis (see, for example, Harba et al. [14], Bonami and Estrade [6], Ponson et al. [25], Roux et al. [27]). For a stochastic process (with $m = n = 1$), the relation (1.1) is called self-similarity (see, for example, Embrechts and Maejima [13], Taqqu [29]). Fractional Brownian motion is the canonical example of a univariate self-similar process, and there are well-established connections between self-similarity and the long-range dependence property of time series (see Samorodnitsky and Taqqu [28], Doukhan et al. [12], Pipiras and Taqqu [24]).

The theory of operator self-similar stochastic processes (namely, $m = 1$) was developed by Laha and Rohatgi [19] and Hudson and Mason [15], see also Chapter 11 in Meerschaert and Scheffler [23]. Operator fractional Brownian motion was studied by Didier and Pipiras [9,10] (see also Robinson [26], Kechagias and Pipiras [17,18] on the related subject of multivariate long range dependent time series). For scalar fields (with $n = 1$), the analogues of fractional Brownian motion and fractional stable motion were studied in depth by Biermé et al. [5], with related work and applications found in Benson et al. [2], Bonami and Estrade [6], Biermé and Lacaux [4], Biermé, Benhamou and Richard [3], Clausel and Vedel [7,8], Meerschaert et al. [22], and Dogan et al. [11]. Li and Xiao [20] proved important results on operator self-similar random vector fields, see Theorem 2.2 below. Baek et al. [1] derived integral representations for Gaussian o.s.s. random fields with stationary increments.

Domain exponents E and range exponents H satisfying (1.1) are not unique in general, due to symmetry. More specifically, the set of domain or range exponents comprises more than one element if and only if the respective set of domain or range symmetries contains a vicinity of the identity. This paper describes the set of possible range exponents H for a given domain exponent E, and conversely, the set of possible domain exponents E for a given range exponent H. In both cases, the difference between two exponents lies in the tangent space of the symmetries. The corresponding result for o.s.s. stochastic processes, the case $m = 1$, was established by Hudson and Mason [15]. In the characterization of the sets of domain or range exponents, the key assumption is that of the existence of a range or a domain exponent, respectively. This allows us to make use of the framework laid out by Hudson and Mason [15], Li and Xiao [20] as well as that of Meerschaert and Scheffler [23], Chapter 5, the latter being more often used for establishing results for domain exponents. In addition, we provide a counterexample showing that the existence of one of the two exponents is a necessary condition for establishing the relation (1.1).

2. Results

This section contains the main results in the paper. All proofs can be found in Section 3.

The domain and range symmetries of X are defined by

$$
\begin{aligned}
G_1^{\mathrm{dom}} &:= \{A \in M(m, \mathbb{R}) : X(At) \simeq X(t)\}, \\
G_1^{\mathrm{ran}} &:= \{B \in M(n, \mathbb{R}) : BX(t) \simeq X(t)\}.
\end{aligned}
\tag{2.1}
$$

For the next proposition, let $GL(k, \mathbb{R})$ be the general linear group on $\mathbb{R}^k$.

Proposition 2.1. *Let $X = \{X(t)\}_{t \in \mathbb{R}^m}$ be a proper nondegenerate random field with values in $\mathbb{R}^n$ such that $X(0) = 0$ a.s. Then, G_1^{ran} is a compact subgroup of $GL(n, \mathbb{R})$, and G_1^{dom} is a compact subgroup of $GL(m, \mathbb{R})$.*