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Openness and weak openness of multiplication in the space of functions of bounded variation

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A B S T R A C T

Let $BV[0,1]$ and $CBV[0,1]$ be spaces of functions of bounded variation and continuous functions of bounded variation, respectively, with the norm $\|f\|_{BV} = |f(0)| + V_0^1(f)$. We shall show that the multiplication is a weakly open operation in $BV[0,1]$ and $CBV[0,1]$. Moreover, the multiplication is an open operation in $BV[0,1]$ with the standard supremum norm. Some other properties of multiplication in function spaces are studied.

1. Preliminaries

Let $C[0,1]$ be the space of all continuous real-valued functions defined on $[0,1]$ with the supremum norm $\|f\| = \sup_{t\in[0,1]} |f(t)|$. There are some natural operations on $C[0,1]$, for example, addition, multiplication, minimum and maximum. In [3,9,10] such operations were investigated. All the operations are continuous but only addition, minimum and maximum are open as mappings from $C[0,1] \times C[0,1]$ to $C[0,1]$.

Remark 1.1 *(Fremlin's example).* [3] In 2004, D.H. Fremlin observed that for $f\colon [0,1] \to \mathbb{R}$, $f(x) = x - \frac{1}{2}$, one has $f^2 \in B^2(f, \frac{1}{2}) \setminus \operatorname{Int} B^2(f, \frac{1}{2})$. Hence multiplication is not an open mapping from $C[0,1] \times C[0,1]$ into $C[0,1]$.

Definition 1.2. [3,2] A map between topological spaces is weakly open if the image of every non-empty open set has a non-empty interior.

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In [3] it is shown that the multiplication in $C[0,1]$ is a weakly open operation. This was generalized in [7] for $C(0,1)$ and in [2] for $C(X)$, where X is an arbitrary interval.

In [6,7] there are considered some properties of multiplication and others operations in the algebra $C(X)$ of real-valued continuous functions defined on a compact topological space X. Properties of the product of open bals and of n open sets in the space of continuous functions on $[0,1]$ are studied in [4] and [5], respectively.

There is an increasing interest in the study of concepts related to the openness and weak openness of natural bilinear maps on certain function spaces. The reason of it may be that the classical Banach open mapping principle is not true for bilinear maps. In [1], the authors show that multiplication from $L^p(X) \times L^q(X)$ onto $L^1(X)$, where (X,μ) is an arbitrary measure space and $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p, q \le \infty$, is an open mapping.

In the paper we study problem of openness and week openness in the spaces $BV[0,1]$ of functions of bounded variation and in the space $CBV[0,1]$ of continuous functions of bounded variation, both defined on $[0,1]$. There are two natural norms in $BV[0,1]$ and $CBV[0,1]$: the supremum norm $\|f\| = \sup\limits_{t\in[0,1]} |f(t)|$ and the norm defined by variation $\|f\|_{BV} = |f(0)| + V_0^1(f)$, where $V_a^b(f) = \sup\limits_{a=t_0<t_1<\ldots<t_n=b} \sum\limits_{i=1}^{n} |f(t_i) - f(t_{i-1})|$ is the variation of f on $[a,b]$. It is worth to mention that $(BV[0,1], \|\ \|_{BV})$ and $(CBV[0,1], \|\ \|_{BV})$ are complete, whereas $(BV, \|\ \|)$ and $(CBV, \|\ \|)$ are not.

We use standard notations, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. $B(f,r)$ and $B_{BV}(f,r)$ denote an open ball centered at f and with the radius r in $BV[0,1]$ or $CBV[0,1]$ with the norm $\|\ \|$ and $\|\ \|_{BV}$, respectively.

2. Multiplication in $CBV[0,1]$

In this section we prove that multiplication in $(CBV[0,1], \|\ \|_{BV})$ and in $(CBV[0,1], \|\ \|)$ is weakly open. Since Fremlin's example still works in $(CBV[0,1], \|\ \|_{BV})$ and in $(CBV[0,1], \|\ \|)$, we know that multiplication is not open neither in $(CBV[0,1], \|\ \|_{BV})$ nor in $(CBV[0,1], \|\ \|)$.

Lemma 2.1. *Let $f \in CBV[0,1]$. For every $\varepsilon > 0$ there exist $b > 0$, $n \in \mathbb{N}$ and intervals $[u_1,v_1]$, $[u_2,v_2]$, $\ldots, [u_n,v_n]$ such that $0 = u_1 < v_1 < u_2 < \ldots < v_{n-1} < u_n < v_n = 1$, $|f(t)| \ge b$ for $t \in [0,1] \setminus \bigcup\limits_{i=1}^{n} [u_i,v_i]$ and $\sum\limits_{i=1}^{n} V_{u_i}^{v_i}(f) < \varepsilon$.*

Proof. Fix $\varepsilon > 0$. Let $\{I_k : k \in K\}$, where K is at most countable, be the family of all connected components of the open set $[0,1] \setminus \{t \in [0,1] : f(x) = 0\}$ and a_k, b_k be the endpoints of I_k. We claim that

$$V_0^1(f) = \sum_{k\in K} V_{a_k}^{b_k}(f). \tag{1}$$

The inequality $\sum\limits_{k\in K} V_{a_k}^{b_k}(f) \le V_0^1(f)$ is obvious. Let $0 = x_0 < x_1 < \ldots < x_m = 1$ be any partition of $[0,1]$ and $L = \{k \in K : x_i \in I_k \text{ for some } i \le m\}$. Define a new partition $\Pi: 0 = y_0 < y_1 < \ldots < y_l = 1$ such that $\{y_j : j \le l\} = \{x_i : i \le m\} \cup \bigcup\limits_{k\in L} \{a_k, b_k\}$. Then

$$\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| \le \sum_{j=1}^{l} |f(y_j) - f(y_{j-1})| \le$$
$$\le \sum_{\{j\colon \exists_{k\in L} y_j, y_{j-1} \in \overline{I_k}\}} |f(y_j) - f(y_{j-1})| \le$$