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A master integral in four parameters

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ABSTRACT

In this paper we consider a master integral in four arbitrary parameters. The integrand involves the logarithmic function and the Gauss hypergeometric function, which in certain special cases the integral reduces to identities involving zeta functions. A relationship will also be created between the integral and Euler sums of arbitrary order and arbitrary argument. Many interesting new specific examples will be highlighted.

1. Introduction and preliminaries

The evaluation of integrals involving integrands with logarithmic and hypergeometric functions can be notoriously difficult to deal with and finding closed form representations of these integrals can be a rare occurrence. Many books and papers have been published on various methods for the evaluation of integrals with hypergeometric or logarithmic functions, see for example [1–4,8–10]. Integrals dealing with the Hurwitz zeta function and Tornheim sums can be seen in [5–7]. A class of logarithmic integrals have also recently been examined in [11]. In particular in this paper we investigate the representation of integrals of the type

$$I\left(m,p,q,t\right)=\int_0^1 \log^m x\ \Lambda\left(p,q,t;x\right)\ dx$$

where

$$\Lambda\left(p,q,t;x\right)=\frac{x^{-\frac{1}{t}}\ \left(1-x^p\right)}{\left(1-x\right)}\,{}_2F_1\left[\left.\begin{matrix}1,\frac{1}{q}\\ 1+\frac{1}{q}\end{matrix}\right|x^p\right]$$

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and ${}_2F_1\left[\begin{matrix}\cdot,\cdot\\ \cdot\end{matrix}\middle| z\right]$ is the Gauss hypergeometric function. We prove that in many cases of the parameters (m,p,q,t) the integral $I(m,p,q,t)$ may be represented in closed form that includes the polygamma and zeta special functions. Finally a generalization of the integral $I(m,p,q,t)$ is given. Let $\mathbb{R}$ and $\mathbb{C}$ denote, respectively the sets of real and complex numbers and let $\mathbb{N} := \{1,2,3,\cdots\}$ be the set of positive integers, with $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. Let $\Gamma(z)$ denote the familiar Euler's gamma function then the digamma (or Psi) function, for $z\in\mathbb{R}$, is defined by

$$\psi(z) := \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)}$$

and is connected to the harmonic number H_z, by $\psi(z+1) = H_z - \gamma$, where γ is the Euler–Mascheroni constant. The Lerch transcendent

$$\Phi(z,t,a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^t} \tag{1.1}$$

is defined for $|z|<1$ and $\Re(a)>0$ and satisfies the recurrence

$$\Phi(z,t,a) = z\,\Phi(z,t,a+1) + a^{-t}.$$

The Lerch transcendent generalizes the Hurwitz zeta function at $z=1$,

$$\Phi(1,t,a) = \zeta(t,a) = \sum_{m=0}^{\infty}\frac{1}{(m+a)^t}$$

and the Polylogarithm, or de Jonquière's function, when $a=1$,

$$Li_t(z) := \sum_{m=1}^{\infty}\frac{z^m}{m^t},\ t\in\mathbb{C}\text{ when } |z|<1;\ \Re(t)>1.$$

Moreover

$$\int_0^1 \frac{Li_t(px)}{x}dx = \begin{cases}\zeta(1+t), & \text{for } p=1\\ (2^{-t}-1)\zeta(1+t), & \text{for } p=-1\end{cases}.$$

A generalized binomial coefficient $\binom{\lambda}{\mu}$ $(\lambda,\mu\in\mathbb{C})$ is defined, in terms of the gamma function, by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)}, \qquad (\lambda,\mu\in\mathbb{C}),$$

which, in the special case when $\mu=n$, $n\in\mathbb{N}_0$, yields

$$\binom{\lambda}{0} := 1 \quad\text{and}\quad \binom{\lambda}{n} := \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n(-\lambda)_n}{n!}\quad (n\in\mathbb{N}),$$

where $(\lambda)_\nu$ is the Pochhammer symbol defined, also in terms of the gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases}1 & (\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\})\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}),\end{cases}$$