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A Dirichlet problem for nonlocal degenerate elliptic operators with internal nonlinearity

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ABSTRACT

We study a Dirichlet problem in the entire space for some nonlocal degenerate elliptic operators with internal nonlinearities. With very mild assumptions on the boundary datum, we prove existence and uniqueness of the solution in the viscosity sense. If we further assume uniform ellipticity then the solution is shown to be classical, and even smooth if both the operator and the boundary datum are smooth.

Contents

1. Introduction

Nonlocal elliptic operators model diffusion processes with long-term interactions. Since Caffarelli and Silvestre introduced the notion of viscosity solutions for these operators [3], one is able to deal with very general classes of nonlocal fully nonlinear elliptic operators, and a theory analogue to the case of second order elliptic equations is established. See for instance the works of Caffarelli–Silvestre [3–5], Kriventsov [8], Jin–Xiong [7], Serra [10] and Yu [11,12].

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In these works a nonlocal fully nonlinear elliptic operator is of the form

$$I[u,x]=\inf_{\alpha}\sup_{\beta}L_{\alpha\beta}u(x), \tag{1.1}$$

where each $L_{\alpha\beta}$ is a linear operator of the form

$$L_{\alpha\beta}u(x)=c_{n,\sigma}\int\delta u(x,y)K_{\alpha\beta}(x,y)dy.$$

Here $c_{n,\sigma}$ is a constant depending on the dimension of the space n, as well as the order of the operator σ, which is always assumed to be in $(1,2)$ in this work. $\delta u(x,y)=u(x+y)+u(x-y)-2u(x)$ is the symmetric difference centered at the point x. $K_{\alpha\beta}(x,y)$ are some kernels comparable to $\frac{1}{|y|^{n+\sigma}}$, which is the kernel for the classical fractional Laplacian $\Delta^{\sigma/2}$.

In a sense, these kernels assign weights to information coming from different locations and directions in the media. Taking the place of coefficient matrices in second order equations, they encode the 'inhomogeneity' and 'anisotropy' of the underlying media.

Since an integral kernel enjoys more 'degrees of freedom' than a matrix, the theory of nonlocal operators allows much richer 'spatial inhomogeneity and anisotropy' in the media. However, for operators as in (1.1), the dependence on $\delta u(x,y)$ is still trivial. To cover possibly different dependence on $\delta u(x,y)$, we propose to study operators of the following form

$$I[u,x]=\int\frac{F(\delta u(x,y))}{|y|^{n+\sigma}}dy, \tag{1.2}$$

where F is an increasing function with $F(0)=0$.

Here the underlying medium is homogeneous and isotropic as the kernel is simply the kernel for fractional Laplacian. But the dependence on δu can take various forms. For instance, $F(t)=10^5t\chi_{|t|<0.01}+t\chi_{0.01<|t|<100}+10^{-5}t\chi_{|t|>100}$ models a diffusion process where one sees strong diffusive effect at 'near equilibrium' points but very weak diffusive effect at 'far from equilibrium' points. For another example, $F(t)=10^5t\chi_{t>0}+10^{-5}t\chi_{t<0}$ models a process where the diffusion is strong at 'convex' points but is weak at 'concave' points.

It is interesting to note that in the limit as $\sigma\to2$, an operator of this form converges, at least formally, to a constant multiple of the Laplacian, the constant being $F'(0)$. This may explain why operators of this form have not received much attention. However, in the case when $\sigma<2$, they do exhibit nontrivial behaviour.

In this paper we study the following Dirichlet problem for this type of operators with 'internal nonlinearity'[1] F:

$$\begin{cases}\int\frac{F(\delta u(x,y))}{|y|^{n+\sigma}}dy=g(x,u-\phi) & \text{in } \mathbb{R}^n\\ u-\phi\to0 & \text{at } \infty.\end{cases}$$

We impose the following conditions throughout the paper on the nonlinearity F, the forcing term g and the boundary datum ϕ:

- $F:\mathbb{R}\to\mathbb{R}$ is a C^1 function which satisfies

$$F(0)=0, \tag{1.3}$$

$$Lip(F)<L_1, \tag{1.4}$$

[1] This term was suggested by Dennis Kriventsov.