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On traveling wave solutions in general reaction–diffusion systems with time delays

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ABSTRACT

We study the existence of traveling wave solutions in a general class of mixed quasi-monotone reaction–diffusion systems with time delays. First, by applying the Schauder Fixed Point Theorem, we prove the existence of a traveling wave solution between classically defined upper and lower solutions. For better applications of the upper–lower solution method on various real-life models, the existence result is further extended under weak form or piecewise smooth upper–lower solutions. In several reaction–diffusion systems with time delays (single-species logistic growth, N-species competition, and ratio-dependent predator–prey with Gompertz growth), we apply our main result to establish the existence of traveling wave solutions flowing towards the positive or coexistent states under reasonable conditions on ecological parameters.

1. Introduction

The method of upper–lower solutions has been widely established during the past decades for existence theory and quantitative analysis on solutions of partial differential solutions [14,21–23]. In recent years, strong research interests are demonstrated for presence, propagation and asymptotic behavior of traveling wave solutions in reaction–diffusion systems modeling natural phenomena [1,2,6,8,11,17,20,25–29]. Among existing literature for multi-equation systems, the existence of traveling wave solutions has been proven for reaction functions that can be transformed into quasi-monotone nondecreasing systems (for example, two-species competition or three-species competition–cooperation) [3,9,10,12,13,15,19]. However, for many real-life models with multiple equations and time delays, the reaction functions possess mixed quasi-

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monotone properties so that it is impossible to transform the system into quasi-monotone non-decreasing type.

In this article, we establish (by the Schauder Fixed Point Theorem) an existence–comparison theorem for traveling wave solutions of a general reaction–diffusion system with time delays where the reaction function $\mathbf{f}(\mathbf{u}, \mathbf{u}_\tau)$ satisfies mixed quasi-monotone properties. Comparing to a previously obtained existence result on delayed systems in [17], our reaction–diffusion system allows more flexibility on the reaction function with presence of both $u_i(x,t)$ and $u_i(x, t-\tau_i)$. The traveling wave solution connects two equilibrium states for limits of the reaction function $\mathbf{f}(\mathbf{u}, \mathbf{u})$ at $\mp\infty$, which is more realistic for various equilibrium states in complex models than the assumption in [17] on traveling wave solution having limit 0 at $-\infty$. Also, in our definition of upper and lower solutions we relax the requirement in [17] that they must have the same limits at both the boundary ends $-\infty$ and $+\infty$. Following the traditional definition [21], we only require appropriate inequalities to hold for boundary conditions, so that it would be much easier to construct upper and lower solutions in complex systems. Furthermore, in the derivation of the existence-comparison theorem we fix a gap on the proof of a compact operator (in Lemma 3.5 of [17]) because Ascoli–Arzela Theorem cannot be applied to $C(R, R^n)$. In order to make the existence-comparison theorem applicable to various real-life models, we show that same result still holds under weak upper–lower solutions which satisfy integral inequalities instead of differential inequalities. Specifically, the existence-comparison result can be extended with a pair of piecewise smooth upper–lower solutions.

We plan the article as follows. In Section 2, for a general reaction–diffusion system with discrete time delays, the upper–lower solutions are defined to satisfy mixed differential inequalities, and their limits at $\mp\infty$ also satisfy corresponding inequalities related to equilibria at the infinities. Through several lemmas, a complete proof of the existence-comparison theorem (Theorem 2.6) is given by applying Schauder's Fixed Point Theorem to a compact operator. We further show that the existence-comparison theorem also holds for weakly coupled upper–lower solutions and piecewise smooth upper–lower solutions. In Section 3, Theorem 2.6 is applied to well-known models with instantaneous and discrete delay effects: the Logistic equation and the N-species competition model of Lotka–Volterra type. Through suitably constructed upper–lower solutions in those reaction–diffusion models with mixed quasi-monotone properties, we obtain conditions for the existence of traveling wave solutions flowing from the extinction state to positive or coexistence state. The construction of those upper–lower solution pairs is based on the classical results on wave solutions for the K.P.P. equation and relevant conditions on ecological parameters. A more complex model, the time-delayed ratio-dependent predator–prey model with Gompertz growth, is discussed in Section 4. In order to prove the existence of traveling wave solutions flowing from the prey-only state $(1,0)$ to the coexistence state (u^*, v^*), we need to show the presence of functions satisfying the defined conditions for upper–lower solutions under mixed quasi-monotone reactions with delays. The construction of those functions is done by applying the extended existence-comparison result in Section 2 with piecewise smooth upper–lower solutions. For all the multi-equation models, we assume different diffusion rates for different population density functions, and find the set of wave speeds (for presence of traveling wave solutions) in terms of the diffusion rates and other ecological parameters.

2. The existence-comparison theorems

We consider the following general reaction–diffusion system with discrete time delays.

$$\frac{\partial \mathbf{u}}{\partial t} = D\Delta \mathbf{u} + \mathbf{f}(\mathbf{u}, \mathbf{u}_\tau) \quad -\infty < x < \infty, \quad t > 0, \tag{2.1}$$

where $\mathbf{u} \equiv \mathbf{u}(x,t) = (u_1(x,t), \ldots, u_n(x,t))$, $\mathbf{f} = (f_1, \ldots, f_n)$, $\mathbf{u}_\tau \equiv \mathbf{u}_\tau(x,t) = \big(u_1(x, t-\tau_1), \ldots, u_n(x, t-\tau_n)\big)$, with $\tau_i \in \mathbb{R}^+$ for $i = 1, \ldots, n$, $D = \operatorname{diag}(d_1, \ldots, d_n)$ with $d_i > 0$ for $i = 1, \ldots, n$. A traveling