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The spherical Radon transform with centers on cylindrical surfaces

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ABSTRACT

Recovering a function from its spherical Radon transform with centers of spheres of integration restricted to a hypersurface is at the heart of several modern imaging technologies, including SAR, ultrasound imaging, and photo- and thermoacoustic tomography. In this paper we study an inversion of the spherical Radon transform with centers of integration restricted to cylindrical surfaces of the form $\Gamma \times \mathbb{R}^m$, where Γ is a hypersurface in $\mathbb{R}^n$. We show that this transform can be decomposed into two lower dimensional spherical Radon transforms, one with centers on Γ and one with a planar center-set in $\mathbb{R}^{m+1}$. Together with explicit inversion formulas for the spherical Radon transform with a planar center-set and existing algorithms for inverting the spherical Radon transform with a center-set Γ, this yields reconstruction procedures for general cylindrical domains. In the special case of spherical or elliptical cylinders we obtain novel explicit inversion formulas. For three spatial dimensions, these inversion formulas can be implemented efficiently by backprojection type algorithms only requiring $\mathcal{O}(\mathtt{N}^{4/3})$ floating point operations, where $\mathtt{N}$ is the total number of unknowns to be recovered. We present numerical results demonstrating the efficiency of the derived algorithms.

1. Introduction

Let Γ be a hypersurface in $\mathbb{R}^n$. In this paper we study the spherical Radon transform with a center-set $\Gamma \times \mathbb{R}^m$ that maps a function $f\colon \mathbb{R}^{n+m} \to \mathbb{R}$ to the spherical integrals

$$(\mathbf{M}_{x,y} f)(x, y, r) := \frac{1}{|\mathbb{S}^{n+m-1}|} \int_{\mathbb{S}^{n+m-1}} f((x, y) + r\omega) \, \mathrm{dS}(\omega) \quad \text{for } (x, y, r) \in \Gamma \times \mathbb{R}^{m+1} .$$

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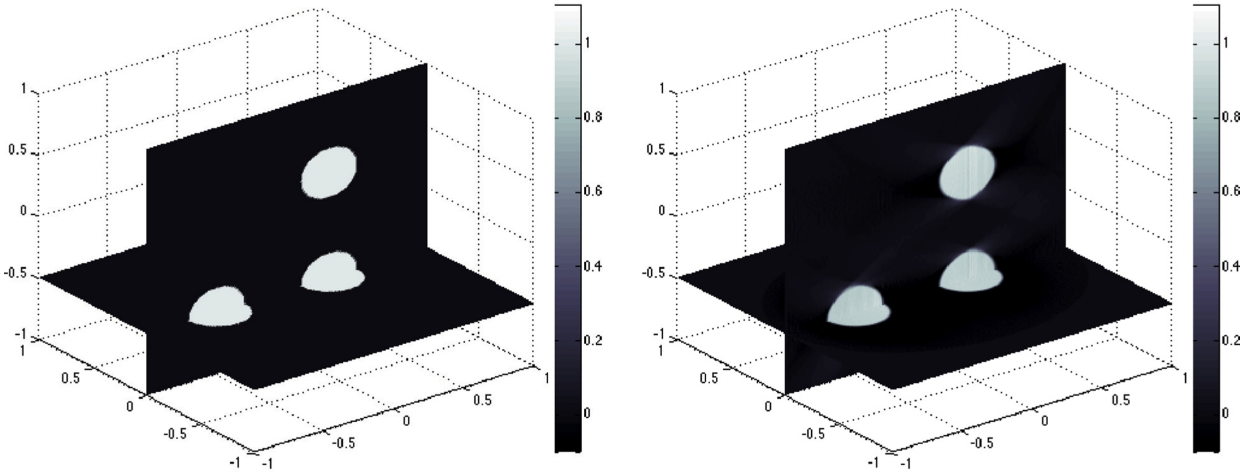

Fig. 1.1. Left: A phantom consisting of a superposition of three balls. Right: 3D reconstruction from its spherical Radon transform on an elliptical cylinder in $\mathbb{R}^3$ using the inversion formula (2) derived in Section 3.

Here $|r|$ and $(x, y) \in \Gamma \times \mathbb{R}^m$ are the radius and the center of the sphere of integration, respectively, $\mathbb{S}^{n+m-1}$ is the unit sphere in $\mathbb{R}^{n+m}$, and $|\mathbb{S}^{n+m-1}|$ is the total surface area of $\mathbb{S}^{n+m-1}$. Note that x, y as subscripts of $(\mathbf{M}_{x,y} f)(x, y, r)$ indicate the variables in which the spherical Radon transform is applied, while in the argument they are placeholders for the actual data points. For example, evaluating the spherical Radon transform at $(x, y, r) = (0, 0, 0)$ gives $(\mathbf{M}_{x,y} f)(0, 0, 0)$. Similar notions will be used for auxiliary transforms introduced below; these suggestive notations are used to facilitate the readability of the manuscript.

Recovering a function from its spherical Radon transform with centers restricted to a hypersurface is crucial for the recently developed thermoacoustic and photoacoustic tomography [23,40]. It is also relevant for other imaging technologies such SAR imaging [20,38] or ultrasound tomography [33].

Explicit inversion formulas for reconstructing f from its spherical Radon transform are of theoretical as well of practical importance. For example, they serve as theoretical basis of backprojection-type reconstruction algorithms frequently used in practice. However, explicit inversion formulas are only known for some special center-sets. Such formulas exist for the case where the center-set is a hyperplane [2,4,5,8,22,29] or a sphere [10,11,24,31,41]. More recently, closed-form inversions have also been derived for the cases of elliptically shaped center-sets (see [3,14,15,30,34,37]), certain quadrics [17,18], oscillatory algebraic sets [35], and corner-like domains [28].

1.1. Main contribution

In this paper we present a general approach for deriving reconstruction algorithms and inversion formulas for the spherical Radon transform on cylindrical surfaces yielding inversion formulas for the center-set $\Gamma \times \mathbb{R}^m$, provided that an inversion formula is known for the center-set Γ. Our approach is based on the observation that the spherical Radon transform with a center-set $\Gamma \times \mathbb{R}^m$ can be written as the composition of a spherical Radon transform with a center-set $\Gamma \subseteq \mathbb{R}^n$ and another spherical Radon transform with a planar center-set in $\mathbb{R}^{m+1}$ (see Theorem 2.3). Recall that inversion formulas for the spherical Radon transform with a planar center-set are well known. Consequently, if an inversion formula is available for the center-set $\Gamma \subseteq \mathbb{R}^n$, then this factorization yields an inversion formula for $\mathbf{M}_{x,y}$. As explicit inversion formulas are in particular known for spherical and elliptical center-sets, we obtain new analytic inversion formulas for spherical or elliptical cylinders (Theorems 3.1 and 3.2). A reconstruction result with one of our inversion formulas for an elliptical cylinder is shown in Fig. 1.1.