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Two-dimensional Riemann problem for Chaplygin gas dynamics in four pieces ☆

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A R T I C L E   I N F O

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A B S T R A C T

In this paper we study the two-dimensional Riemann problem for Chaplygin gas with initial data being four states, and focus on the wave structures with fewer restriction on the initial data. For the solution made up of elementary waves and the density ρ satisfying $0 < \rho < +\infty$, we mainly analyse the necessary and sufficient condition on the initial data as the self-similar solution can be constructed uniquely. In particular, if the initial densities are identical, we obtain a necessary and sufficient condition concretely and construct the solution explicitly.

1. Introduction

In this paper, we consider the two-dimensional Riemann problem of the Euler system for the Chaplygin gas. The isentropic Euler system of the inviscid adiabatic compressible flow can be written as follows:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \vec{u}) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho \vec{u} \bigotimes \vec{u}) + \nabla p = 0, \end{cases} \tag{1.1}$$

where ρ, $\vec{u} = (u, v)$ and p denote density, velocity and pressure respectively. System (1.1) with state equation

$$p = g(s) - \frac{f(s)}{\rho}$$

or simplified state equation [3]

$$p = -\frac{1}{\rho}, \tag{1.2}$$

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is called a Chaplygin gas. Here s is the specific entropy and is a constant for an isentropic gas.

Chaplygin gas proposed as a cosmological model provides hints for what could happen for real gases. It is not only a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics [3,11], but also can be viewed as a one-dimensional version of the Born–Infeld system [1] which is a nonlinear modification of the Maxwell equations to solve the electrostatic divergence generated by point particles in classical Electrodynamics [6].

For the one-dimensional Riemann problem of Chaplygin gas, the previous works on it show that there are some wave structures which differ from the polytropic gas such as concentration of mass. Y. Brenier first found the concentrate effect δ-wave of Chaplygin gas [2]. And the generalised solutions with δ initial data is obtained [12] under the generalised Rankine–Hugoniot conditions and the entropy condition. Without the mass concentration limit in the initial data, the Riemann solution is proved to be stable under the local small perturbation of the Riemann initial data even when the initial perturbed density depends on the parameter [8].

The two-dimensional Riemann problem, initial data with four piecewise constant states, is studied in [5,7,9], and with three piecewise constant states is studied in [4,13]. The case that initial data result in one wave coming from infinity in each direction of discontinuity was considered in [5,7] where the possible structures of solution including δ wave were conjectured. The case that initial data result in two waves coming from infinity in each direction of discontinuity is studied and classified in [4] under the assumption that the initial data are relatively small in some sense.

In this paper, we are mainly concerned with the solvability condition of the two-dimensional Riemann problem with fewer restrictions on the initial data within the classical theory on weak solution [10]. Namely, we study the necessary and sufficient condition for the solvability of the Riemann problem of (1.1), (1.2) in the sense that the density ρ does neither vanish nor become infinity since vacuum ($\rho = 0$) also causes singularity in view of (1.2). Two waves are allowed to come from infinity in each initial discontinuity direction. Without loss of generality, we consider the simplified state equation (1.2). Immediately, we have the sonic speed

$$c := \sqrt{\frac{\partial p}{\partial \rho}} = \frac{1}{\rho}. \tag{1.3}$$

Initially, the (x, y) plane is divided by $\theta_i = \frac{i}{2}\pi$ into regions $\Omega_i : \theta_{i-1} < \theta < \theta_i$, the state of gas in Ω_i is

$$U_i = (u_i, v_i, c_i). \tag{1.4}$$

Assume the initial data satisfy

$$u_i \cos\theta_i + v_i \sin\theta_i = u_{i+1}\cos\theta_i + v_{i+1}\sin\theta_i, \tag{1.5}$$

$i = 1, 2, 3, 4$. Or equivalently,

$$v_1 = v_2, \quad u_2 = u_3, \quad v_3 = v_4, \quad u_4 = u_1. \tag{1.6}$$

In the (u, v) plane, these four points (u_i, v_i) locate at the vertexes of a rectangle. For simplicity and concreteness, assume that these points form a square.

Let $d = |u_1 - u_2| = |v_2 - v_3|$, i.e. the length of one side of the square. Then there are four different cases as shown in Fig. 1.

We call the kind of interactions of elementary waves resulting from the initial discontinuity as **the first interaction**, and the kind induced by the first interactions as **the second interaction**. Then there are four first interactions for each case in Fig. 1. The second interaction makes the wave structure more complicated. We