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Density of discrete semigroup flows

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ABSTRACT

We generalize the notions of upper and lower density on semigroups to semigroup flows. We then show that when the Strong Følner Condition holds, there exist upper densities on the flow satisfying classical properties such as partition regularity of positive sets, finite additivity on translates, and invariance under semigroup action. We also introduce the concept of Følner density of a flow and show that under a certain condition, it is multiplicative on products.

1. Introduction

In this paper, our goal is to generalize the concept of density of means from semigroups to flows.

It is well known that the Stone–Čech compactification of a semigroup is a right topological semigroup, and that the Stone–Čech compactification of a flow (S, X) is a flow that is continuous in the first variable (i.e. in the semigroup variable; see [9]). We begin by reviewing this structure, approaching the topic via ultrafilters.

The concepts of upper and lower density are well known for the natural numbers, and widely used in areas of number theory and Ramsey theory. Hindman and Strauss generalized these in their papers [7] and [8], to an arbitrary semigroup. In order to give these densities the pleasant properties that exist for density on the natural numbers, they used Følner nets and amenability of semigroups. The key concept required here is that of Følner conditions – combinatorial conditions characterizing amenability of groups that were first introduced by the eponymous Følner in [6]. Namioka, in [11], generalized these conditions to characterize amenability of semigroups. Further work was conducted on this topic in [15], [1], [16] and [10]. Sakai introduced amenability of a flow in [14], and generalized Følner conditions to flows in [13]. Using these, we are able to generalize Hindman and Strauss' results on density of means to flows. We refer the reader to [5] and [12] for references on amenability.

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Section 2 covers some basic preliminaries on flows, and background on the Følner conditions. In Section 4, we build the concepts of upper and lower density of a flow (S, X) with respect to a net of finite subsets of X. We then show that if (S, X) satisfies the Strong Følner Condition, we can define an upper density on it that is additive on translations of sets, and invariant under the action of S – i.e. satisfies the pleasant properties held by the density on the natural numbers. We also introduce Følner density for (S, X) and show that this notion of density is well-behaved with similar properties. Furthermore, we show that when a certain condition is satisfied, this density acts as expected on finite products of flows, where the density of a finite product of sets is equal to the product of the densities of the individual sets. Finally, in Section 5, we conclude with some open problems on density of means.

2. Preliminaries and notation

For any set X, we denote by $|X|$ the cardinality of X. We define $\mathscr{P}(X)$ to be the power set of X and $\mathscr{P}_f(X)$ to be all finite non-empty subsets of X. If X is a topological space, for each $A \subset X$, $cl(A)$ will denote the closure of A in X.

Given a non-empty set X, let $m(X)$ be the space of bounded real-valued functions on X equipped with the supremum norm. Let F be a linear subspace of $m(X)$ that contains all the constant functions. A **mean** on F is a continuous linear functional $M : F \to \mathbb{R}$ such that $M(\chi_X) = 1$ and $\|M\| = 1$. We shall denote the set of means on $m(X)$ by $\mathscr{M}(X)$. It is a well known fact that $\mathscr{M}(X)$ is non-empty, convex and w^*-compact as a subset of $m(X)^*$ (see [5]).

A **left semigroup flow** (or **left flow**) is a triple (S, X, p) where S is a semigroup, X is a non-empty set and $p : S \times X \to X$ is a map that satisfies $p(st, x) = p(s, p(t, x))$, for all $s, t \in S$, and $x \in X$. Such a function p is called a **left action** of S on X. Note that in this paper we do not assume continuity of the action map p in our definition. We will use the shorthand notation (S, X) for a flow (S, X, p), and shorten $p(s, x)$ to sx for each $s \in S$ and $x \in X$.

Let (S, X) be a flow. Given a subsemigroup T of S, and $Y \subset X$, we call (T, Y) a **subflow** if $tY \subset Y$, for each $t \in T$. We say an element $s \in S$ is S**-cancellable** if the map $X \to X$, $x \mapsto sx$ is injective. If all the elements of S are S-cancellable, we say that (S, X) is an S**-cancellative** flow. For each $s \in S$, we define the S**-translation operator** $L_s : m(X) \to m(X)$ to be given by $[L_s(f)](x) = f(sx)$, for each $x \in X$ and $f \in m(X)$. We define (S, X) to be **amenable** if there is a mean M on $m(X)$ that is S-invariant, i.e. $M(f) = M(L_s f)$, for all $s \in S$, $f \in m(X)$. We denote S-invariant means on X by $\mathscr{M}_l(X)$.

(S, X) is said to satisfy the **Strong Følner Condition (SFC)** if for any $\varepsilon > 0$, and $F \in \mathscr{P}_f(S)$, there exists $A \in \mathscr{P}_f(X)$, such that $|A \backslash sA| \le \varepsilon |A|$ for each $s \in F$. On the other hand, (S, X) is said to satisfy the **Følner Condition (FC)** if for any $\varepsilon > 0$, and $F \in \mathscr{P}_f(S)$, there exists $A \in \mathscr{P}_f(X)$, such that $|sA \backslash A| \le \varepsilon |A|$ for each $s \in F$.

Remark 2.1. Note that for any $A \in \mathscr{P}_f(X)$, and $s \in S$,

$$|sA \backslash A| = |sA \backslash (A \cap sA)| = |sA| - |A \cap sA| \le |A| - |A \cap sA| = |A \backslash sA|,$$

and thus, the Strong Følner Condition implies the Følner Condition.

Suppose (S, X) is a semigroup flow and $\{F_\alpha\}_{\alpha \in A} \subset \mathscr{P}_f(x)$ is a net. Then we call $\{F_\alpha\}_{\alpha \in A}$ a **Følner net**, if for every $s \in S$,

$$\lim_{\alpha \in A} \frac{|sF_\alpha \Delta F_\alpha|}{|F_\alpha|} = 0$$

By Remark 2.1, this is equivalent to