J. Math. Anal. Appl. 448 (2017) 707–721

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A mortar finite volume method for a fractured model in porous media ☆

Shuangshuang Chen, Hongxing Rui *

School of Mathematics, Shandong University, Jinan, 250100, China

A R T I C L E   I N F O

Article history:
Received 8 August 2016
Available online 16 November 2016
Submitted by G. Chen

Keywords:
Fracture model
The mortar finite volume scheme
Non-conforming meshes
Error estimates
Numerical experiments

A B S T R A C T

In this paper, we consider a mortar finite volume method for a fractured model of flow in porous media. In this model, the permeability coefficients are variable between the fracture and the surrounding porous media. A finite volume method based on Raviart–Thomas elements combined with the mortar technique of domain decomposition is presented, in which sub-domains are triangulated independently and the meshes do not match at interfaces. The great advantage of the method is avoiding solving the saddle-point problem, since the numerical scheme is just related to the pressure p, and the velocity $\boldsymbol{u}$ can be expressed by p. We also prove error estimates of order h on the discrete H^1 norm between the exact solution p and the mortar finite volume solution P and the $(L^2)^2$ norm between $\boldsymbol{u}$ and $\boldsymbol{U}$. Finally, numerical experiments have been performed to show the consistency of the convergence rates with the theoretical analysis.

1. Introduction

The finite volume method is widely used in numerical simulation of elliptic, parabolic or hyperbolic problems due to the local conservation property and good accuracy. It was discussed based on the finite differences using rectangular meshes in the early years, and then it was extended to general triangular grids combined with the finite element method. Now it can be used on arbitrary geometries, structured or unstructured meshes and still maintains the discrete local mass conservation, so the finite volume method has become the most important technique in several engineering fields. The mathematical framework of detailed descritization and convergence analysis of the finite volume method for various partial differential equations are shown in [7]. And we also refer to [8,9] for elliptic problems and [10] for coupling system.

A finite volume method for the convection–diffusion equation on an opened two-dimensional domain was presented by Herbin in 1995 [11]. The 4-point numerical scheme using triangular meshes was defined,

☆ The work is supported by the National Natural Science Foundation of China Grant No. 91330106, 11671233.

* Corresponding author.
E-mail address: author.hxrui@sdu.edu.cn (H. Rui).

http://dx.doi.org/10.1016/j.jmaa.2016.11.019

the numerical solution was approximated by a finite volume piecewise constant and the error estimate for the discrete L^2 norm was analyzed. Then more general non-structured meshes were used for this scheme in [12]. Let us also mention [16], in which elliptic problems with Dirichlet, Neumann, or Robin boundary conditions were all studied using unstructured grids. In 2001 [6], an approximate gradient of this solution was introduced, and the convergence of this reconstructed gradient was also considered.

The mortar finite element method as a coupling technique is important in solving problems with complicated domains and interacting processes, since it allows us to use independent grids, form different discretization methods, or even solve different mathematic problems in different parts of the domain. The mortar finite element method has become an attractive nonconforming technique due to its flexibility and the parallel computation property. A mortar finite volume element scheme for elliptic equations on non-matching meshes was studied in [6], and in the paper, the discretization was based on the Petrov–Galerkin method and an optimal order error estimate in energy norm was obtained. Then a mortar finite volume method using the Crouzeix–Raviart non-conforming elements was constructed in [5]. In our paper, in view of the domain decomposition technique, we discuss the numerical method when the whole domain contains several sub-domains which are gridded independently, so the meshes will not match at interfaces. The basic idea of the mortar method is to enforce the interface continuity conditions in a weak sense. There are a large variety of approaches of replacing the stranger continuity condition by a weak one, such as [13,3]. We also can refer to [2,4] for details.

We combine the finite volume method proposed by Herbin with the mortar technique in this paper, the mortar finite volume method. It is an attractive choice in numerical reservoir simulation because it holds the flexibility of the mortar method and the local conservation property of finite volume method simultaneously. The model that we are concerned about is fluid flow in porous media with a known fracture, and in this model the permeability coefficients are variable between the fracture and surrounding porous media [14, 1]. It is reasonable to use fine grids in the domains with high permeability since the Darcy velocity and pressure will change rapidly. We use mortar idea to treat the fracture as a sub-domain and triangulate it independently. Meshes need not match across interfaces, which allows to make local and adaptive changes of grids on any subset without modeling the grids of others. The finite volume scheme is discussed in every sub-domain separately. And then the major work is to find matching conditions, i.e., the so-called mortar conditions, to ensure the weak continuity of approximation functions on the interfaces. The mortar conditions are presented in our paper to keep the approximations having the same L^2 projections on the interfaces between two adjacent sub-domains.

In addition to the flexibility of independent triangulations of fracture and surrounding domains, the great advantage of the mortar finite volume method proposed is avoiding solving the saddle-point problem. We just need to form the numerical scheme about the pressure p, and then the velocity $\boldsymbol{u}$ in the Raviart–Thomas finite element space [15] with the smallest order can be expressed by p. It will greatly reduce the computational cost. The rest of the paper is organized as follows. In section 2 we introduce the fractured model and do some preparations for the discretization. In section 3 we propose the mortar conditions across interfaces and present the mortar finite volume scheme. In section 4 convergence theory is proved. Error estimates for the discrete H^1 norm between the exact solution p and the mortar finite volume solution P and the $(L^2)^2$ norm between $\boldsymbol{u}$ and $\boldsymbol{U}$ on the non-matching triangular grids are all first-order accuracy. In section 5 some numerical experiments are implemented to test the performance of the mortar finite volume method, and results show the consistency of the convergence rates with the theoretical analysis.

2. Description of the problem

Let Ω be a bounded domain in $\mathbb{R}^2$, with boundary Γ. We assume that flow in Ω is governed by Darcy's equation relating the gradient of the pressure p to the Darcy velocity $\boldsymbol{u}$, and a conservation equation. Suppose that we have a Dirichlet boundary condition on Γ,