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Boundary measurement matrices for directed networks on surfaces

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A B S T R A C T

Franco, Galloni, Penante, and Wen have proposed a boundary measurement map for a graph on any closed orientable surface with boundary. We consider this boundary measurement map which takes as input an edge weighted directed graph embedded on a surface and produces on element of a Grassmannian. Computing the boundary measurement requires a choice of fundamental domain. Here the boundary measurement map is shown to be independent of the choice of fundamental domain Also, a formula for the Plücker coordinates of the element of the Grassmannian in the image of the boundary measurement map is given. The formula expresses the Plücker coordinates as a rational function which can be combinatorially described in terms of paths and cycles in the directed graph.

1. Introduction

The totally nonnegative Grassmannian was defined by Postnikov [9] and can be studied using edge weighted planar graphs embedded on a disk. These edge weighted planar graphs and the totally nonnegative Grassmannian are connected to the physics of scat-

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tering amplitudes and $\mathcal{N} = 4$ super Yang–Mills [1]. In the context of physics, the edge weighted planar graphs are usually called "on-shell diagrams." A key element of Postnikov's study of the totally nonnegative Grassmannian is the boundary measurement map which produces an element of the totally nonnegative Grassmannian for any edge weighted directed graph embedded in the disk. Under a mild hypothesis on the graph, Talaska [10] gives a formula for the Plücker coordinates of the element of the totally nonnegative Grassmannian corresponding to a given graph. In [3,4] a boundary measurement map for graphs on more general surfaces is proposed with the hopes of going beyond the "planar limit" of $\mathcal{N} = 4$ super Yang–Mills.

The definition of the boundary measurement map will be given later in this section, and in defining the boundary measurement we must make a choice of how to represent our directed graph in the plane. The boundary measurement map turns out to be independent of this choice as we will see in Section 2. We will show in Section 3 how boundary measurement map can be obtained by signing the edges of a directed graph. This technique of signing edges will allow us to unify two formulas of Talaska [10,11]. A formula for the Plücker coordinates corresponding to the boundary measurement map is given in Section 4. In Section 5 we will show that the signs used in Section 3 are unique up to the gauge action.

1.1. *Weighted path matrices*

Let $N = (V, E)$ be a directed graph with finite vertex set V and finite edge set E. This means an edge $e \in E$ is an ordered pair $e = (i, j)$ for $i, j \in V$. If $e = (i, j)$ then the edge e is said to be directed from vertex i to vertex j. For each edge $e \in E$ of N we associate a formal variable x_e. We will work in $\mathbb{R}[[x_e : e \in E]]$ the ring of formal power series in the variables $\{x_e\}_{e \in E}$ with coefficients in $\mathbb{R}$. As in [9], we will use the term *directed network* $N = (V, E)$ to refer to the directed graph $N = (V, E)$ along with edge weights $\{x_e\}_{e \in E}$.

A *path* is a finite sequence of edges $P = (e_1, e_2, \cdots, e_l)$ where $e_k = (i_{k-1}, i_k)$ for $1 \leq k \leq l$. If $P = (e_1, e_2, \cdots, e_l)$ where $e_1 = (i_0, i_1)$ and $e_l = (i_{l-1}, i_l)$, then P is said to be a path from i_0 to i_l. The path P is said to be *self avoiding* if $i_k \neq i_{k'}$ for $k \neq k'$. The path P is called a *cycle* if $i_0 = i_l$, and we say the cycle is a *simple cycle* when $i_k = i_{k'}$ if and only if $k = k'$ or $\{k, k'\} = \{0, l\}$. We use the notation $P : i \rightsquigarrow j$ to denote a path from i to j. When $P = (e_1, e_2, \cdots, e_l)$ we let

$$\mathrm{wt}(P) = \prod_{i=1}^{l} x_{e_i}$$

denote the *weight* of the path P.

We order our vertex set V and consider the $V \times V$ *weighted path matrix* $M = M(N, \{x_e\}_{e \in E})$ with entries given by

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