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A Plancherel measure associated to set partitions and its limit

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ABSTRACT

In recent years increasing attention has been paid on the area of supercharacter theories, especially to those of the upper unitriangular group. A particular supercharacter theory, in which supercharacters are indexed by set partitions, has several interesting properties, which make it object of further study. We define a natural generalization of the Plancherel measure, called superplancherel measure, and prove a limit shape result for a random set partition according to this distribution. We also give a description of the asymptotical behavior of two set partition statistics related to the supercharacters. The study of these statistics when the set partitions are uniformly distributed has been done by Chern, Diaconis, Kane and Rhoades.

1. Introduction

Let p be a prime number, q a power of p, and $\mathbb{K}$ the finite field of order q and characteristic p. Consider $U_n = U_n(\mathbb{K})$ to be the group of upper unitriangular matrices with entries in $\mathbb{K}$, it is known that the description of conjugacy classes and irreducible

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characters of U_n is a wild problem, in the sense described, for example, by Drodz in [13]. To bypass the issue, André [2] and Yan [25] set the foundations of what is now known as "supercharacter theory" (in the original works it was called "basic character theory"). The idea is to meld together some irreducible characters and conjugacy classes (called respectively supercharacters and superclasses), in order to have characters which are easy enough to be tractable but still carry information of the group. In particular, one obtains a smaller character table, which is required to be a square matrix. As an application, in [3], Arias-Castro, Diaconis and Stanley described random walks on U_n utilizing only the supercharacter table (usually the complete character table is required). In [12], Diaconis and Isaacs formalized the axioms of supercharacter theory, generalizing the construction from U_n to algebra groups.

Among the various supercharacter theories for U_n a particular nice one, hinted in [1] and described by Bergeron and Thiem in [6], has the property that the supercharacters take integer values on superclasses. This is particularly interesting because of a result of Keller [17], who proves that for each group G there exists a unique finest supercharacter theory with integer values. Although it is not yet known if Bergeron and Thiem's theory is the finest integral one, it has remarkable properties which make it worth of a deeper analysis. In this theory the supercharacters of U_n are indexed by set partitions of $\{1, \dots, n\}$ and they form a basis for the Hopf algebra of superclass functions. This Hopf algebra is isomorphic to the algebra of symmetric functions in noncommuting variables. Moreover, the supercharacter table decomposes as the product of a lower triangular matrix and an upper triangular matrix.

In the theory introduced by Bergeron and Thiem, the characters depend on the following three statistics defined for a set partition π of $[n]$:

- $d(\pi)$, the number of arcs of π;
- $\dim(\pi)$, that is, the sum $\sum \max(B) - \min(B)$, where the sum runs through the blocks B of π;
- $\operatorname{crs}(\pi)$, the number of crossings of π.

More precisely, we have that if χ^π is the supercharacter associated to the set partition π then the dimension is $\chi^\pi(1) = q^{\dim(\pi) - d(\pi)}$ and $\langle \chi^\pi, \chi^\pi \rangle = q^{\operatorname{crs}(\pi)}$.

In the setting of probabilistic group theory one is interested in the study of statistics of the "typical" irreducible representation of the group. A natural probability distribution is the uniform distribution; in [10] and [11] Chern, Diaconis, Kane and Rhoades study the statistics dim and crs for a uniform random set partition, proving formulas for the moments of $\dim(\pi)$ and $\operatorname{crs}(\pi)$ and, successively, a central limit theorem for these two statistics. These results imply that, for a uniform random set partition π of n,

$$\dim(\pi) - d(\pi) = \frac{\alpha_n - 2}{\alpha_n} n^2 + O_P\left(\frac{n}{\alpha_n}\right), \qquad \operatorname{crs}(\pi) = \frac{2\alpha_n - 5}{4\alpha_n^2} n^2 + O_P\left(\frac{n}{\alpha_n}\right),$$

where α_n is the positive real solution of $ue^u = n+1$, so that $\alpha_n = \log n - \log\log n + o(1)$.