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A combinatorial understanding of lattice path asymptotics

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We provide a combinatorial derivation of the exponential growth constant for counting sequences of lattice path models restricted to the quarter plane. The values arise as bounds from analysis of related half planes models. We give explicit formulas, and the bounds are provably tight. The strategy is easily generalizable to cones in higher dimensions, and has implications for random generation.

1. Introduction

Lattice path models have enjoyed a sustained popularity in mathematics over the past century, owing in part to their simplicity and ease of analysis, but also their wide applicability both in mathematics, physics, and chemistry. The basic enumerative question is

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to determine the number of walks of a given length in a given model. The past fifteen years have seen many interesting developments in the asymptotic and exact enumeration of lattice models, with new techniques coming from computer algebra, complex analysis and algebra. A first approximation to this value is the *exponential growth constant*, also called the *connective constant*, which itself carries combinatorial and probabilistic information. For example, it is directly related to the limiting free energy in statistical mechanical models.

Here, a lattice path model is defined by the steps that are allowed, and the region to which the walks are restricted (generally cones and strips). Particular focus has been on small step models (where the steps are a subset of $\{0, \pm 1\}^2$) restricted to $\mathbb{Z}^2_{\geq 0}$, and general approaches versus resolution of individual cases. For example, several distinct strategies for asymptotic enumeration have recently emerged. Fayolle and Raschel [10] have determined expressions for the growth constant for small step models using boundary value problem techniques. Recast as diagonals, techniques of analytic combinatorics of several variables apply to some of the models with D-finite generating functions [15, 16]. Also, the important sub-class of excursions is well explored via the probability work of Denisov and Wachtel [7, Section 1.5]. Bostan, Raschel and Salvy [5] made their results explicit in the enumeration context. Most of these asymptotic results are obtained with machinery which does not sustain a clear underlying combinatorial picture.

Many of these results exclude *singular* models: A two dimensional model is singular if the support of the step set is contained in a half plane. Many singular models are either trivial or reduce to a problem in a lower dimension. Singular models are considered in [17,14].

This paper provides a formula for an upper bound on the growth constant of the counting sequence for lattice models restricted to a convex cone, with intuitive combinatorial interpretations of intermediary computations. Our formula is most explicit in the case of nonsingular 2-dimensional walks restricted to the first quadrant, but is valid *for all models*. The core of our strategy is based on the following basic observation:

In any lattice path model, the set of walks restricted to the first quadrant is a subset of the walks restricted to some half plane which contains that quadrant. Consequently, for any fixed length, the number of walks in that half plane is an upper bound for the number of walks in the quarter plane.

Bounds on walks in half planes are readily computable, for example using the results of Banderier and Flajolet [1]. Remarkably, we are able to give *tight* bounds on the growth constant by considering all of the half planes that contain the quarter plane. Furthermore, our bounds are insightfully tight in that they give a single simple combinatorial interpretation of the multiple cases treated by Fayolle and Raschel [10]. Our one idea *unifies* their cases, which depend on various parameters of the model. Our approach also applies to singular models. We use only the elementary calculus observation that a minimum of a real valued differentiable function f with domain D must occur either at the boundary of D or at a critical point $\tau \in D$ satisfying $f'(\tau) = 0$. This strategy remains