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Two-color balanced affine urn models with multiple drawings

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A B S T R A C T

We study a class of balanced urn schemes on balls of two colors (say white and black). At each drawing, a sample of size $m \geq 1$ is taken out from the urn, and ball addition rules are applied. We consider these multiple drawings under sampling with or without replacement. We further classify ball addition matrices according to the structure of the expected value into affine and nonaffine classes. We give a necessary and sufficient condition for a scheme to be in the affine subclass, for which we get explicit results for the expected value and second moment of the number of white balls after n steps and an asymptotic expansion of the variance. Moreover, we uncover a martingale structure. This unifies several earlier works focused on special cases of urn models with multiple drawings [5,6,17,19,20,22] as well as the special case of sample size $m = 1$. The class is parametrized by Λ, specified by the ratio of the two eigenvalues of a "reduced" ball replacement matrix and the sample size. We categorize the class into small-index urns ($\Lambda < \frac{1}{2}$), critical-index urns ($\Lambda = \frac{1}{2}$), and large-index urns ($\Lambda > \frac{1}{2}$), and triangular urns. We obtain central limit theorems for small- and critical-index urns and prove almost-sure convergence for

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triangular and large-index urns. Moreover, we discuss the moment structure of large-index urns and triangular urns.

1. Introduction

Urn schemes are simple, useful and versatile mathematical tools for modeling many evolutionary processes in diverse applications such as algorithmics, genetics, epidemiology, physics, engineering, economics, networks (social and other types), and many more. Modeling via urns is centuries old, but perhaps the earliest contributions in the flavor commonly called Pólya urns (the subject of the present paper) are [7,8]. In the first of these two classics, urns were meant to model contagion. In the second, urns were intended to model the diffusion of gases. Many Pólya urn models useful for numerous applications were added later on. In fact, they are too many (literally hundreds) to be listed individually. The sources [13,16] are classic surveys listing many of these applications; see also [18], where two chapters are devoted to applications in algorithmics and biosciences.

While the term Pólya urn refers to a vast variety of schemes, there is a common thread among most of them. Urns of the classic flavor on two colors (say white and black) evolve in the following way. At the beginning, time zero, the urn contains a certain number of white and black balls. Thereafter, evolution of the urn occurs in discrete time steps. At every step, a ball is chosen at random from the urn. The color of the ball is inspected, then the ball is reinserted in the urn. According to the color of the sampled ball, other balls are added/removed following certain rules—if we have chosen a white ball, we put in the urn a white balls and b black balls, but if we have chosen a black ball, we put in the urn c white balls and d black balls. The values $a, b, c, d \in \mathbb{Z}$ are fixed. The urn model is specified by the 2×2 ball replacement matrix $\mathbf{M} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. One is interested in the number of white balls W_n and black balls B_n after n draws.

1.1. *Pólya urn models with multiple drawings*

In classic Pólya urns, one ball is sampled at each unit of (discrete) time. The present work is devoted to the study of a generalization of the Pólya urn model, where *multiple* balls are drawn at each discrete time step, their colors are inspected, then the sample is reinserted in the urn. Additions and deletions take place according to the drawn sample (multiset). Such urn models recently received attention in the literature, see for example [5,6,14,17,19–22]. The addition/removal of balls depends on the combinations of colors of the drawn balls. We use the notation $\{W^k B^{m-k}\}$ to refer to a sample of size m containing k white balls and $m - k$ black balls. Specifically, we draw $m \geq 1$ balls and add/remove white and black balls according to the multiset $\{W^k B^{m-k}\}$ of observed