

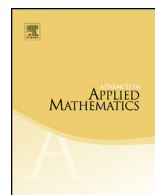


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# The sequence of open and closed prefixes of a Sturmian word <sup>☆</sup>

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## ABSTRACT

A finite word is closed if it contains a factor that occurs both as a prefix and as a suffix but does not have internal occurrences, otherwise it is open. We are interested in the *oc-sequence* of a word, which is the binary sequence whose  $n$ -th element is 0 if the prefix of length  $n$  of the word is open, or 1 if it is closed. We exhibit results showing that this sequence is deeply related to the combinatorial and periodic structure of a word. In the case of Sturmian words, we show that these are uniquely determined (up to renaming letters) by their *oc-sequence*. Moreover, we prove that the class of finite Sturmian words is a maximal element with this property in the class of binary factorial languages. We then discuss several aspects of Sturmian words that can be expressed through this sequence. Finally, we provide a linear-time algorithm that computes the *oc-sequence* of a finite word, and a linear-time algorithm that reconstructs a finite Sturmian word from its *oc-sequence*.

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## 1. Introduction

In a recent paper with M. Bucci [5], the first two authors dealt with trapezoidal words (a generalization of finite Sturmian words), also with respect to the property of being closed or open. Let  $\Sigma$  be a finite nonempty set (the alphabet). A (finite) word  $w = w[1]w[2] \cdots w[n]$  with  $w[i] \in \Sigma$  is *closed* (also known as *periodic-like* [6]) if it contains a factor that occurs both as a prefix and as a suffix but does not have internal occurrences, otherwise it is *open*. For example, the words  $abca$ ,  $ababa$  and  $aabaab$  are closed — any word of length 1 is closed, the empty word being a factor that occurs both as a prefix and as a suffix but does not have internal occurrences; the words  $ab$ ,  $aab$  and  $abaa$ , instead, are open.

Given a finite or infinite word  $w = w[1]w[2] \cdots$ , the sequence  $oc(w)$  of open/closed prefixes of  $w$ , that we refer to as the *oc-sequence* of  $w$ , is the binary sequence  $c(1)c(2) \cdots$  whose  $n$ -th element is 1 if the prefix of  $w$  of length  $n$  is closed, 0 if it is open. For example, if  $w = abcab$ , then  $oc(w) = 10011$ .

A question that arises naturally is whether it is possible to reconstruct a word (up to renaming letters) from its oc-sequence. This is not true in general, even when the alphabet is binary. For example, the words  $aaba$  and  $aabb$  are not isomorphic (i.e., one cannot be obtained from the other by renaming letters), yet they have the same oc-sequence 1100. As a first result of this paper, we show that if a word is known to be Sturmian, then it can be reconstructed (up to renaming letters) from its oc-sequence. That is, Sturmian words are characterized by their oc-sequences. Moreover, we prove that the class of finite Sturmian words is a maximal element with this property in the class of binary factorial languages.

In [5], the authors investigated the structure of the sequence  $oc(F)$  of the Fibonacci word  $F$ . They proved that the lengths of the runs (maximal subsequences of consecutive equal elements) in  $oc(F)$  form the doubled Fibonacci sequence. We prove in this paper that this doubling property holds for every standard Sturmian word, and describe the sequence  $oc(w)$  of a standard Sturmian word  $w$  in terms of the *semicentral* prefixes of  $w$ , which are the prefixes of the form  $u_nxyu_n$ , where  $x, y$  are letters and  $u_nxy$  is an element of the standard sequence of  $w$ . As a consequence, we show that the word  $ba^{-1}w$ , obtained from a standard Sturmian word  $w$  starting with letter  $a$  by replacing the first letter with a  $b$ , can be written as the infinite product of the words  $(u_n^{-1}u_{n+1})^2$ ,  $n \geq 0$ . Since the words  $u_n^{-1}u_{n+1}$  are reversals of standard words, this induces an infinite factorization of  $ba^{-1}w$  in squares of reversed standard words.

We then show how the oc-sequence of a standard Sturmian word of slope  $\alpha$  is related to the continued fraction expansion of  $\alpha$ , both in terms of the convergents and of the continuants of  $\alpha$ .

Finally, we provide a linear-time algorithm that computes the oc-sequence of a finite word, and a linear-time algorithm that reconstructs a finite Sturmian word from its oc-sequence.

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