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Eulerian polynomials and descent statistics

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ABSTRACT

We prove several identities expressing polynomials counting permutations by various descent statistics in terms of Eulerian polynomials, extending results of Stembridge, Petersen, and Brändén. Additionally, we find q-exponential generating functions for q-analogues of these descent statistic polynomials that also keep track of the inversion number or inverse major index. We also present identities relating several of these descent statistic polynomials to refinements of type B Eulerian polynomials and flag descent polynomials by the number of negative letters of a signed permutation. Our methods include permutation enumeration techniques involving noncommutative symmetric functions, the modified Foata–Strehl action, and a group action of Petersen on signed permutations. Notably, the modified Foata–Strehl action yields an analogous relation between Narayana polynomials and the joint distribution of the peak number and descent number over 231-avoiding permutations, which we also interpret in terms of binary trees and Dyck paths.

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1. Introduction

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation in $\mathfrak{S}_n$, the set of permutations of $[n] = \{1, 2, \ldots, n\}$, which are called *n-permutations*.[1] Also, let $|\pi|$ be the length of π—so that $|\pi| = n$ whenever $\pi \in \mathfrak{S}_n$—and let $\mathfrak{S} := \bigcup_{n=0}^{\infty} \mathfrak{S}_n$. We say that $i \in [n-1]$ is a *descent* of an n-permutation π if $\pi_i > \pi_{i+1}$. Every permutation can be uniquely decomposed into a sequence of maximal increasing consecutive subsequences—or equivalently, maximal consecutive subsequences containing no descents—which we call *increasing runs*. For example, the descents of $\pi = 85712643$ are 1, 3, 6, and 7, and the increasing runs of π are 8, 57, 126, 4, and 3.

Let $\operatorname{des}(\pi)$ denote the number of descents of π. Then it is clear that the number of increasing runs of π is $\operatorname{des}(\pi)+1$ when $|\pi| \geq 1$, i.e., when π is nonempty. The polynomial

$$A_n(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\operatorname{des}(\pi)+1}$$

for $n \geq 1$ is called the nth *Eulerian polynomial*. We set $A_0(t) = 1$ by convention.[2] The exponential generating function for Eulerian polynomials is well known:

$$\sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!} = \frac{1-t}{1-te^{(1-t)x}}.$$

The Eulerian polynomials have a rich history and appear in many contexts in combinatorics; see [18] for a detailed exposition.

The Eulerian polynomials are closely related to the distribution of other descent statistics: permutation statistics that depend only on the descent set and length of a permutation. Specifically, it is known that polynomials counting permutations by various descent statistics—the number of peaks, left peaks, and biruns—can be expressed in terms of Eulerian polynomials.

- We say that i (where $2 \leq i \leq n-1$) is a *peak* of $\pi = \pi_1\pi_2\cdots\pi_n$ if $\pi_{i-1} < \pi_i > \pi_{i+1}$, and let $\operatorname{pk}(\pi)$ be the number of peaks of π. For example, the peaks of $\pi = 85712643$ are 3 and 6, and so $\operatorname{pk}(\pi) = 2$. The peak polynomials

  $$P_n^{\operatorname{pk}}(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\operatorname{pk}(\pi)+1}$$

  are related to the Eulerian polynomials by the identity

[1] By convention, we take $\mathfrak{S}_0$ to consist of only the empty word.

[2] In general, for all polynomials defined in this paper counting permutations by various statistics—such as $P_n^{\operatorname{pk}}(t)$, $P_n^{(\operatorname{pk},\operatorname{des})}(y,t)$, etc.—we set the 0th polynomial to be 1 by convention.