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On the complexity of the word problem for automaton semigroups and automaton groups

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A B S T R A C T

In this paper, we study the word problem for automaton semigroups and automaton groups from a complexity point of view. As an intermediate concept between automaton semigroups and automaton groups, we introduce automaton-inverse semigroups, which are generated by partial, yet invertible automata. We show that there is an automaton-inverse semigroup and, thus, an automaton semigroup with a PSpace-complete word problem. We also show that there is an automaton group for which the word problem with a single rational constraint is PSpace-complete. Additionally, we provide simpler constructions for the uniform word problems of these classes. For the uniform word problem for automaton groups (without rational constraints), we show NL-hardness. Finally, we investigate a question asked by Cain about a better upper bound for the length of a word on which two distinct elements of an automaton semigroup must act differently.
A detailed listing of the contributions of this paper can be found at the end of this paper.

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1. Introduction

Traditionally, algebraic structures have been presented by specifying generators and relations. There is, however, another kind of presentation based on finite Mealy automata. Algorithmically, it has a remarkable advantage: it guarantees decidability of the word problem for the generated groups and semigroups. This stands in contrast to the traditional way of presenting such structures: even if the set of generators and the set of relations are both finite, one can (finitely) present a group with undecidable word problem (a classical result due to Boone and Novikov from the mid 50s). This means in particular that the word problem is not decidable for every group and every semigroup. Thus, not every group or semigroup is an *automaton group* or *automaton semigroup*, i.e. generated by an automaton. While at first sight this seems to be a severe limitation, automaton groups and semigroups have proven to have deep connections with many areas of mathematics from the theory of profinite groups and complex dynamics to theoretical computer science.

Many examples of groups with interesting properties are in fact automaton groups. The most notable example is probably Grigorchuk's first example of a group of intermediate (i.e. super-polynomial but sub-exponential) growth (see e.g. [12] for an accessible introduction). The existence of such a group answers the classical Milnor problem. This group is also an example of an infinite, finitely generated torsion group (Burnside problem) and an amenable non-elementary amenable group (von Neumann problem). Its discovery led to the development of a new and exiting branch of research in group theory by Grigorchuk and others: the study of finitely presented groups that act (transitively) on each level of an infinite regular rooted tree (see e.g. [16]). This is just a different way of describing automaton groups.

Decidability of the word problem for automaton groups is most often proven by giving a (deterministic) exponential time algorithm. The idea behind this algorithm is that one can give an exponential upper bound on the level of the tree on which two distinct group elements act differently. For some special sub-classes of automaton groups, there is a better upper bound on how many paths in the tree need to be checked (see e.g. [2] for polynomial-activity automata, [3] for Hanoi Tower groups, [16] for contracting automaton groups). However, as Steinberg notes in [24], there is a more straightforward – at least from a computer science perspective – nondeterministic algorithm requiring linear space (and, thus, also yielding a deterministic exponential time algorithm). This algorithm leads Steinberg to the question whether there is an automaton group with a PSpace-complete word problem.

In this paper, we give some partial answers to Steinberg's question. We do not only consider automaton groups but also automaton semigroups. This class is a generalization where the generating automata are not required to be invertible. While it is less studied than automaton groups, it has attracted quite some interest recently (for example in [1,7–9,11] or in the work of Cain [6], and Cain and Brough [4,5]). Indeed, Gillibert showed that the finiteness problem for automaton semigroups is undecidable [10] while an