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On the score sheets of a round-robin football tournament ☆

Bogdan Ichim [a], Julio José Moyano-Fernández [b,*]

[a] *Simion Stoilow Institute of Mathematics of the Romanian Academy, Research Unit 5, P.O. Box 1-764, 014700 Bucharest, Romania*
[b] *Universitat Jaume I, Campus de Riu Sec, Departamento de Matemáticas & Institut Universitari de Matemàtiques i Aplicacions de Castelló, 12071 Castellón de la Plana, Spain*

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ABSTRACT

The set of (ordered) score sheets of a round-robin football tournament played between n teams together with the pointwise addition has the structure of an affine monoid. In this paper we study (using both theoretical and computational methods) the most important invariants of this monoid, namely the Hilbert basis, the multiplicity, the Hilbert series and the Hilbert function.

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* Corresponding author.
E-mail addresses: bogdan.ichim@imar.ro (B. Ichim), moyano@uji.es (J.J. Moyano-Fernández).

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1. Introduction

Football matches are nowadays ubiquitous, stirring up deep passions in the society. Within the several procedures for selecting a winner from among many contestants, round-robin tournaments are widely extended. A round-robin tournament is a competition in which each contestant meets all other contestants in turn. Famous examples of this kind of tournaments are the group stages of the FIFA World Cup, the UEFA European Football Championship and the UEFA Champions League (each group contains 4 teams). Goal difference (calculated as the number of goals scored minus the number of goals conceded) may be used to separate teams that are at the same level on points (according to the specific rules of a given tournament).

We consider score sheets of round-robin tournaments as tables showing the number of goals each team scores any other. Then, the problem of counting all possible score sheets (with the total number of goals fixed) of a certain tournament becomes naturally a problem for combinatorics and statistics. Moreover, we consider *ordered* score sheets in the sense that the teams are ordered by the total number of goals scored. While counting score sheets in general is an easy problem, the effective counting of ordered score sheets turns out to be a quite challenging problem for which a possible approach is presented in this paper. We would like point out that the counting of score sheets is a relevant issue in the theory of weighted logic and statistical games; for instance, see the book of Epstein [12, Chapter 9] which deals with score sheets of round-robin tournaments recording the number of points obtained by teams.

Round-robin tournaments may also be seen as directed graphs (see [14] or [15]). We do not follow this approach here, our presentation is based on monoid theory.

In a recent paper D. Zeilberger [13] deduced a formula for counting of the number of ways for the four teams of the group stage of the FIFA World Cup to each have r goals *for* and r goals *against*. This is nothing but counting the number of 4×4 magic squares with line-sums equal to r under the restriction of all diagonal entries to be 0—the problem without this restriction was treated by R. P. Stanley [21, Proposition 4.6.19]. He also changes the number of teams in a computational experiment, see his web-page [22].

Inspired by these results, we present in this paper a systematic study of the score sheets. We consider the set of (ordered) score sheets of a round-robin football tournament played between n teams. Since it turns out that this set has a natural structure of an affine monoid, say $\mathcal{M}_n$, we focus our attention on the most important invariants of the monoid — that are by no means easy to compute in general. More precisely, we study the Hilbert basis, the multiplicity, the Hilbert series and the Hilbert function of $\mathcal{M}_n$.

We note that, from a practical point of view, the most important numerical invariants are the Hilbert series and the Hilbert function, since they may be used for counting the number of ordered score sheets that may result from a round-robin football tournament. However, it turns out that they are also the most difficult to compute.

This paper is structured as follows. In Section 2 we review some standard facts on rational cones and affine monoids. For details we refer the reader to Bruns and Gubeladze