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Volume inequalities for sections and projections of Wulff shapes and their polars ☆

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Let $1 \leq k \leq n$. Sharp volume inequalities for k-dimensional sections of Wulff shapes and dual inequalities for projections are established. As their applications, several special Wulff shapes are investigated.

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1. Introduction

Throughout, all Borel measures are understood to be nonnegative and finite. A convex body in $\mathbb{R}^n$ is a compact convex set containing the origin in its interior. The polar body of a convex body K is given by $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}$, where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. We use $\|\cdot\|$ to denote the Euclidean norm on $\mathbb{R}^n$. When A is a compact convex set in $\mathbb{R}^n$, we write $|A|$ for the volume of A in the appropriate subspace. Let $\operatorname{supp}\nu$ denote the support of a measure ν and let P_H be the orthogonal projection onto a subspace H of $\mathbb{R}^n$.

Volume estimates for sections of convex bodies in $\mathbb{R}^n$ are not easy, even in specific cases. For the cube $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$, Hensley [10] first showed that if H is a hyperplane of $\mathbb{R}^n$ then $|H \cap Q_n|$ lies between 1 and 5, and conjectured that the upper bound is at most $\sqrt{2}$. This conjecture was solved by Ball [1,2], who also settled the more general case of k-dimensional sections.

The example of the regular simplex is much more complicated. Webb [25] proved that the maximal central hyperplane section is the one containing $n-1$ vertices and the centroid. The question about the minimal central hyperplane section has not been completely solved yet. Brzezinski [7] proved a lower bound which differs from the conjectured minimal volume by a factor of approximately 1.27. For general k-dimensional sections, these questions were recently considered by Dirksen [8]. Other examples, such as ℓ_p^n-balls [5,6,20], complex cubes [22] and non-central sections of cubes [21], have also been investigated.

In this paper, we will study sections and projections of more general convex bodies than cubes and simplices. The main objects we consider are Wulff shapes [23], which were introduced by Wulff in 1901. Nowadays, it is an important notion in convex geometric analysis (see, e.g., [23]).

Definition. Suppose that ν is a Borel measure on S^{n-1} and that f is a positive, bounded, and measurable function on S^{n-1}. The *Wulff shape* $W_{\nu,f}$ determined by ν and f is defined by

$$W_{\nu,f} := \{x \in \mathbb{R}^n : x \cdot u \leq f(u) \quad \text{for all} \quad u \in \operatorname{supp}\nu\}. \tag{1.1}$$

The measure ν is said to be *even* if it assumes the same value on antipodal sets. When ν and f are both even, then $W_{\nu,f}$ is origin-symmetric. It is easy to see that $W_{\nu,f}$ is always convex and may be unbounded. In order for a k-dimensional subspace H of $\mathbb{R}^n$, to guarantee that $|W_{\nu,f}|$ and $|H \cap W_{\nu,f}|$ are finite, we consider Wulff shapes determined by measures ν which are isotropic and f-centered with respect to H. A Borel measure ν on S^{n-1} is called *isotropic* if

$$\int_{S^{n-1}} u \otimes u d\nu(u) = I_n, \tag{1.2}$$