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Computing Gaussian & exponential measures of semi-algebraic sets

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A B S T R A C T

We provide a numerical scheme to approximate as closely as desired the Gaussian or exponential measure $\mu(\mathbf{\Omega})$ of (not necessarily compact) basic semi-algebraic sets $\mathbf{\Omega} \subset \mathbb{R}^n$. We obtain two monotone (non-increasing and non-decreasing) sequences of upper and lower bounds $(\overline{\omega}_d)$, $(\underline{\omega}_d)$, $d \in \mathbb{N}$, each converging to $\mu(\mathbf{\Omega})$ as $d \to \infty$. For each d, computing $\overline{\omega}_d$ or $\underline{\omega}_d$ reduces to solving a semidefinite program whose size increases with d. Some preliminary (small dimension) computational experiments are encouraging and illustrate the potential of the method. The method also works for any measure whose moments are known and which satisfies Carleman's condition.

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1. Introduction

Given a basic semi-algebraic set

$$\mathbf{\Omega} = \{ \mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0, \; j = 1, \dots, m \}, \tag{1.1}$$

for some polynomials $(g_j) \subset \mathbb{R}[\mathbf{x}]$, we want to compute (or more precisely, approximate as closely as desired) $\mu(\mathbf{\Omega})$ for the standard Gaussian measure on $\mathbb{R}^n$:

$$\mu(B) = \frac{1}{(2\pi)^{n/2}} \int_B \exp(-\frac{1}{2}\|\mathbf{x}\|^2)\, d\mathbf{x}, \qquad \forall\, B \in \mathcal{B}(\mathbb{R}^n), \tag{1.2}$$

and the standard exponential measure on the positive orthant $\mathbb{R}^n_+$:

$$\mu(B) = \frac{1}{(2\pi)^{n/2}} \int_B \exp(-\sum_{i=1}^n x_i)\, d\mathbf{x}, \qquad \forall\, B \in \mathcal{B}(\mathbb{R}^n_+). \tag{1.3}$$

This problem is "canonical" as for a non-centered Gaussian with density $\exp(-(\mathbf{x} - m)^T \mathbf{Q}(\mathbf{x} - m))$ (for some real symmetric positive definite matrix $\mathbf{Q}$) or for an exponential measure with density $\exp(-\sum_i \lambda_i x_i)$ one may always reduce the problem to the above one by an appropriate change of variable. Indeed after this change of variable the new domain is again a basic semi-algebraic set of the form (1.1).

Computing $\mu(\mathbf{\Omega})$ has applications in Probability & Statistics where the Gaussian measure plays a central role. In full generality with sets $\mathbf{\Omega}$ as general as (1.1), it is a difficult and challenging problem even for rectangles $\mathbf{\Omega}$, e.g.:

$$\frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{a_1} \int_{-\infty}^{a_2} \exp(-(x^2 + y^2 - 2\rho\, xy)/2(1-\rho^2))\, dx\, dy, \tag{1.4}$$

in small dimension like $n = 2$ or $n = 3$. Indeed, citing A. Genz [10]: *bivariate and trivariate probability distributions computation are needed for many statistics applications … high quality algorithms for bivariate and trivariate probability distribution computations have only more recently started to become available.* For instance, Genz [10] describes techniques with high accuracy results for bivariate and trivariate "rectangles" (1.4) using sophisticated techniques to integrate Plackett's formula. Again those efficient techniques takes are very specific as they take advantage of Plackett's formula available for (1.4). Interestingly, a (complicated) formula in closed form is provided in Chandramouli and Ranganathan [6] via the characteristic function method.

The case of ellipsoids $\mathbf{\Omega}$ has been investigated in the pioneering work of Ruben [20], Kotz et al. [13,14] when studying the distribution of random variables that are quadratic forms of independent normal variables. Even in small dimension it has important applications in Astronautics where for instance $\mu(\mathbf{\Omega})$ can model the probability of collision