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Free cumulants, Schröder trees, and operads

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ABSTRACT

The functional equation defining the free cumulants in free probability is lifted successively to the noncommutative Faà di Bruno algebra, and then to the group of a free operad over Schröder trees. This leads to new combinatorial expressions, which remain valid for operator-valued free probability. Specializations of these expressions give back Speicher's formula in terms of noncrossing partitions, and its interpretation in terms of characters due to Ebrahimi-Fard and Patras.

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1. Introduction

1.1. Functional equations and combinatorial Hopf algebras

Recent works on certain functional equations involving reversion of formal power series have revealed that the appropriate setting for their combinatorial understanding involved a series of noncommutative generalizations, ending up as an equation in the group of an operad.

Roughly speaking, this amounts to first interpreting the equation in the Faà di Bruno Hopf algebra, lifting it to its noncommutative version, and then replacing the constant term by a new indeterminate, giving rise to tree expanded series.

This approach to functional inversion has been initiated in [16–18], and then extended in [13] to deal with the conjugacy equation for formal diffeomorphisms.

Free probability provides other examples of functional equations with a combinatorial solution. The relation between the moments and the free cumulants of a single random variable is just a functional inversion, which can be treated combinatorially by the formalism of [13]. However, the case of several random variables is classically formulated as a triangular system of equations which is solved by Möbius inversion over the lattice of noncrossing partitions [19,20].

We shall see that this system can be encoded by a single equation in the group of an operad. This version encompasses the case of an operator valued probability. The solution arises as a sum over reduced plane trees which reduces to Speicher's solution in the scalar case. Also, our functional equation gives back that of Ebrahimi-Fard and Patras [7], which interpret the series of the moments as a character of a Hopf algebra, and that of the free cumulants as an infinitesimal character, both related by a dendriform exponential. We have here a similar structure.

1.2. Classical probability

The free cumulants k_n of a probability measure μ on $\mathbb{R}$ are defined (see *e.g.*, [19]) by means of the generating series of its moments m_n

$$M_\mu(z) := \int_{\mathbb{R}} \frac{\mu(dx)}{z-x} = z^{-1} + \sum_{n\geq 1} m_n z^{-n-1} \tag{1}$$

as the coefficients of its compositional inverse

$$K_\mu(z) := M_\mu(z)^{\langle -1\rangle} = z^{-1} + \sum_{n\geq 1} k_n z^{n-1}\,. \tag{2}$$

The formal series M_μ is called the Cauchy transform of μ, and K_μ its $\mathcal{R}$-transform. By abuse of language, we shall also say that K_μ is the $\mathcal{R}$-transform of M_μ.