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A general Beurling–Helson–Lowdenslager theorem on the disk

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The classical Beurling–Helson–Lowdenslager theorem characterizes the shift-invariant subspaces of the Hardy space H^2 and of the Lebesgue space L^2. In this paper, which is self-contained, we define a very general class of norms α and define spaces H^α and L^α. We then extend the Beurling–Helson–Lowdenslager invariant subspace theorem. The idea of the proof is new and quite simple; most of the details involve extending basic well-known $\|\cdot\|_p$-results for our more general norms.

1. Introduction

Why should we care about invariant subspaces? In finite dimensions all of the structure theorems for operators can be expressed in terms of invariant subspaces. For example the statement that every $n \times n$ complex matrix T is unitarily equivalent to an upper

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triangular matrix is equivalent to the existence of a chain $M_0 \subset M_1 \subset \cdots \subset M_n$ of T-invariant linear subspaces with $\dim M_k = k$ for $0 \leq k \leq n$. Since every upper triangular normal matrix is diagonal, the preceding result yields the spectral theorem. A matrix is similar to a single Jordan block if and only if its set of invariant subspaces is linearly ordered by inclusion, so the Jordan canonical form can be completely described in terms of invariant subspaces. In [3] L. Brickman and P.A. Fillmore describe the lattice of all invariant subspaces of an arbitrary matrix.

In infinite dimensions, where we consider closed subspaces and bounded operators, even the existence of one nontrivial invariant subspace remains an open problem for Hilbert spaces. If T is a normal operator with a $*$-cyclic vector, then, by the spectral theorem, T is unitarily equivalent to the multiplication operator M_z on $L^2(\sigma(T), \mu)$, i.e.,

$$(M_z f)(z) = z f(z),$$

where μ is a probability Borel measure on the spectrum $\sigma(T)$ of T. In this case von Neumann proved that if a subspace W that is invariant for M_z and for $M_z^* = M_{\bar{z}}$, then the projection P onto W is in the commutant of M_z, which is the maximal abelian algebra $\{M_\varphi : \varphi \in L^\infty(\mu)\}$. Hence there is a Borel subset E of $\sigma(T)$ such that $P = M_{\chi_E}$, which implies $W = \chi_E L^2(\mu)$. It follows that if T is a *reductive* normal operator, i.e., every invariant subspace for T is invariant for T^*, then all invariant subspaces of T have the form $\chi_E L^2(\mu)$. In [8] D. Sarason characterized the (M_z, μ) that are reductive; in particular, when $T = M_z$ is unitary (i.e., $\sigma(T) \subset \mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$), then M_z is reductive if and only if Haar measure m on $\mathbb{T}$ is not absolutely continuous with respect to μ. When $\sigma(T) = \mathbb{T}$ and $\mu = m$ is Haar measure on $\mathbb{T}$, then M_z on L^2 is the bilateral shift operator.

If T is the restriction of a normal operator to an invariant subspace with a cyclic vector e, then there is a probability space $L^2(\sigma(T), \mu)$ such that T is unitarily equivalent to M_z restricted to $P^2(\mu)$, the closure of the polynomials in $L^2(\mu)$, and where e corresponds to the constant function $1 \in P^2(\mu)$. If $\sigma(T) = \mathbb{T}$ and $\mu = m$, then $P^2(\mu)$ is the classical Hardy space H^2 and M_z is the unilateral shift operator.

In infinite dimensions the first important characterization of all the invariant subspaces of a non-normal operator, the unilateral shift, was due to A. Beurling [2] in 1949. His result was extended by H. Helson and D. Lowdenslager [6] to the bilateral shift operator, which is a non-reductive unitary operator.

In this paper, suppose $\mathbb{D}$ is the unit disk in the complex plane $\mathbb{C}$, and m is Haar measure (i.e., normalized arc length) on the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. We let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, respectively, denote the sets of real numbers, integers, and positive integers. Since $\{z^n : n \in \mathbb{Z}\}$ is an orthonormal basis for L^2, we see that M_z is a *bilateral shift* operator. The subspace H^2 which is the closed span of $\{z^n : n \geq 0\}$ is invariant for M_z and the restriction of M_z to H^2 is a *unilateral shift* operator. A closed linear subspace W of L^2 is *doubly invariant* if $zW \subseteq W$ and $\bar{z}W \subseteq W$. Since $\bar{z}z = 1$ on $\mathbb{T}$, W is doubly