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# Lattice paths and the $q$-ballot polynomials

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Two statistics with respect to "upper-corners" and "lower-corners" are introduced for lattice paths. The corresponding refined generating functions are shown to be closely related to the $q$-ballot polynomials that extend the well-known Narayana polynomials and Catalan numbers.



## 1. Introduction and outline

Lattice path enumeration has wide applications to combinatorics (cf. [1,7,24]) and has been extensively studied (see Mohanty [22] and Narayana [23] for example). Roughly

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speaking, a lattice path in $\mathbb{N}_0^2$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $\mathbb{N}$ beinging the set of natural numbers, from the origin to $(m,n)$ is a walk consisting of horizontal steps $H = (1,0)$ and vertical steps $V = (0,1)$. Denote by $\mathbf{P}(m,n)$ the set of lattice paths running from $(0,0)$ to $(m,n)$ without restrictions. It is well-known that the lattice paths $\mathbf{P}(m,n)$ are counted by the binomial coefficient $\binom{m+n}{n}$. When the paths are weighted by the area function, the corresponding generating function becomes the $q$-binomial coefficient $\left[{m+n \atop n}\right]$. If considering the subset $\mathcal{P}(m,n) \subset \mathbf{P}(m,n)$, the lattice paths from the origin to $(m,n)$ remaining below the line $x = y$ (without crossing this diagonal), we reach the ballot number (cf. Carlitz [9])

$$|\mathcal{P}(m,n)| = \frac{1+m-n}{1+m}\binom{m+n}{n}.$$

However, there exists no closed expression of the generating function for $\mathcal{P}(m,n)$ weighted by area parameter. Therefore, it is necessary to investigate generating functions with respect to different statistics.

Recall that a path $L \in \mathbf{P}(m,n)$ can be expressed by the consecutive points

$$L = \{Q_k\}_{k=0}^{m+n} \subset \mathbb{N}_0^2 : \quad \begin{cases} \text{where} \quad Q_0 = (0,0) \quad \text{and} \quad Q_{m+n} = (m,n), \\ Q_{k+1} - Q_k \in \{H, V\} \quad \text{for} \quad 0 \le k < m+n. \end{cases}$$

Just like peaks and valleys in Dyck paths, a point $Q_k \in L$ is said to be a "upper-corner" of $L$, denoted by "$\ulcorner(Q_k)$", if $Q_k - Q_{k-1} = V$ and $Q_{k+1} - Q_k = H$, i.e., $Q_k$ is a "turning point" of $L$ from a vertical step $V$ to an horizontal step $H$. Similarly, a point $Q_k \in L$ is called a "lower-corner" of $L$, denoted by "$\lrcorner(Q_k)$", if $Q_k - Q_{k-1} = H$ and $Q_{k+1} - Q_k = V$, i.e., $Q_k$ is a "turning point" of $L$ from an horizontal step $H$ to a vertical step $V$. Hence, a lattice path $L \in \mathbf{P}(m,n)$ can be represented by its "upper-corners"

$$\ulcorner(L) = \big\{\ulcorner_1(x_1,y_1), \ulcorner_2(x_2,y_2), \cdots, \ulcorner_k(x_k,y_k)\big\}$$

with their coordinates subject to the following conditions

$$\ulcorner(L) : \quad \begin{Bmatrix} 0 \le x_1 < x_2 < \cdots < x_k < m \\ 0 < y_1 < y_2 < \cdots < y_k \le n \end{Bmatrix}.$$