

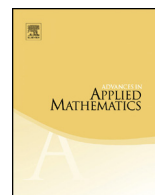


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Lattice paths and the q -ballot polynomials



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ABSTRACT

Two statistics with respect to “upper-corners” and “lower-corners” are introduced for lattice paths. The corresponding refined generating functions are shown to be closely related to the q -ballot polynomials that extend the well-known Narayana polynomials and Catalan numbers.

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1. Introduction and outline

Lattice path enumeration has wide applications to combinatorics (cf. [1,7,24]) and has been extensively studied (see Mohanty [22] and Narayana [23] for example). Roughly

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speaking, a lattice path in \mathbb{N}_0^2 , where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with \mathbb{N} being the set of natural numbers, from the origin to (m, n) is a walk consisting of horizontal steps $H = (1, 0)$ and vertical steps $V = (0, 1)$. Denote by $\mathbf{P}(m, n)$ the set of lattice paths running from $(0, 0)$ to (m, n) without restrictions. It is well-known that the lattice paths $\mathbf{P}(m, n)$ are counted by the binomial coefficient $\binom{m+n}{n}$. When the paths are weighted by the area function, the corresponding generating function becomes the q -binomial coefficient $\left[\begin{smallmatrix} m+n \\ n \end{smallmatrix} \right]$. If considering the subset $\mathcal{P}(m, n) \subset \mathbf{P}(m, n)$, the lattice paths from the origin to (m, n) remaining below the line $x = y$ (without crossing this diagonal), we reach the ballot number (cf. Carlitz [9])

$$|\mathcal{P}(m, n)| = \frac{1 + m - n}{1 + m} \binom{m + n}{n}.$$

However, there exists no closed expression of the generating function for $\mathcal{P}(m, n)$ weighted by area parameter. Therefore, it is necessary to investigate generating functions with respect to different statistics.

Recall that a path $L \in \mathbf{P}(m, n)$ can be expressed by the consecutive points

$$L = \{Q_k\}_{k=0}^{m+n} \subset \mathbb{N}_0^2 : \begin{cases} \text{where } Q_0 = (0, 0) \text{ and } Q_{m+n} = (m, n), \\ Q_{k+1} - Q_k \in \{H, V\} \text{ for } 0 \leq k < m + n. \end{cases}$$

Just like peaks and valleys in Dyck paths, a point $Q_k \in L$ is said to be a “upper-corner” of L , denoted by “ $\Uparrow(Q_k)$ ”, if $Q_k - Q_{k-1} = V$ and $Q_{k+1} - Q_k = H$, i.e., Q_k is a “turning point” of L from a vertical step V to an horizontal step H . Similarly, a point $Q_k \in L$ is called a “lower-corner” of L , denoted by “ $\lrcorner(Q_k)$ ”, if $Q_k - Q_{k-1} = H$ and $Q_{k+1} - Q_k = V$, i.e., Q_k is a “turning point” of L from an horizontal step H to a vertical step V . Hence, a lattice path $L \in \mathbf{P}(m, n)$ can be represented by its “upper-corners”

$$\Gamma(L) = \{\Gamma_1(x_1, y_1), \Gamma_2(x_2, y_2), \dots, \Gamma_k(x_k, y_k)\}$$

with their coordinates subject to the following conditions

$$\Gamma(L) : \left\{ \begin{array}{l} 0 \leq x_1 < x_2 < \dots < x_k < m \\ 0 < y_1 < y_2 < \dots < y_k \leq n \end{array} \right\}.$$

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