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Character formulas and descents for the hyperoctahedral group

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A general setting to study a certain type of formulas, expressing characters of the symmetric group $\mathfrak{S}_n$ explicitly in terms of descent sets of combinatorial objects, has been developed by two of the authors. This theory is further investigated in this paper and extended to the hyperoctahedral group B_n. Key ingredients are a new formula for the irreducible characters of B_n, the signed quasisymmetric functions introduced by Poirier, and a new family of matrices of Walsh–Hadamard type. Applications include formulas for natural B_n-actions on coinvariant and exterior algebras and on the top homology of a certain poset in terms of the combinatorics of various classes of signed permutations, as well as a B_n-analogue of an equidistribution theorem of Désarménien and Wachs.

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1. Introduction

One of the main goals of combinatorial representation theory, as described in the survey article [9], is to provide explicit formulas which express the values of characters of interesting representations as weighted enumerations of nice combinatorial objects. Perhaps the best known example of such a formula is the Murnaghan–Nakayama rule [29, page 117] [42, Section 7.17] for the irreducible characters of the symmetric group $\mathfrak{S}_n$.

Several such formulas of a more specific type, expressing characters of the symmetric groups and their Iwahori–Hecke algebras in terms of the distribution of the descent set over classes of permutations, or other combinatorial objects, have been discovered in the past two decades. The prototypical example is Roichman's rule [35] for the irreducible characters of $\mathfrak{S}_n$ (and the corresponding Hecke algebra), where the enumerated objects are either Knuth classes of permutations, or standard Young tableaux. Other notable examples include the character of the Gelfand model (i.e., the multiplicity-free sum of all irreducible characters) [3], characters of homogeneous components of the coinvariant algebra [2], Lie characters [24], characters of Specht modules of zigzag shapes [23], characters induced from a lower ranked exterior algebra [17] and k-root enumerators [37], determined by the distribution of the descent set over involutions, elements of fixed Coxeter length, conjugacy classes, inverse descent classes, arc permutations and k-roots of the identity permutation, respectively. The formulas in question evaluate these characters by $\{-1, 0, 1\}$-weighted enumerations of the corresponding classes of permutations, where exactly the same weight function appears in all summations. Some new examples are given in Sections 7–8 of this paper.

An abstract framework for this phenomenon, which captures all aforementioned examples, was proposed in [6]. Characters which are expressed by such formulas (see Definition 3.1) are called *fine characters*, and classes which carry them are called *fine sets*. It was shown in [6] that the equality of two fine characters is equivalent to the equidistribution of the descent set over the corresponding fine sets. This implies, in particular, the equivalence of classical theorems of Lusztig–Stanley in invariant theory [40] and Foata–Schützenberger in permutation statistics [22]. Furthermore, it was shown in [6] that fine sets can be characterized by the symmetry and Schur-positivity of the associated quasisymmetric functions. For the latter, the reader is referred to Gessel–Reutenauer's seminal paper [24].

This paper investigates this setting further and provides a nontrivial extension to the hyperoctahedral group B_n. Section 3.1 gives a more explicit version of the main result of [6]. This version (Theorem 3.2) states that a given $\mathfrak{S}_n$-character is carried by a fine set $\mathcal{B}$ if and only if its Frobenius characteristic is equal to the quasisymmetric generating function of the descent set over $\mathcal{B}$. To extend this result to the group B_n, suitable signed analogues of the concepts of fine characters and fine sets have to be introduced (see Definition 3.5) and suitable signed analogues of the fundamental quasisymmetric functions, namely those defined and studied by Poirier [32], are employed. For the former task, a signed analogue of the concept of descent set is used (see Section 2.2) and