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Difference dimension quasi-polynomials

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We consider Hilbert-type functions associated with difference (not necessarily inversive) field extensions and systems of algebraic difference equations in the case when the translations are assigned some integer weights. We will show that such functions are quasi-polynomials, which can be represented as alternating sums of Ehrhart quasi-polynomials associated with rational conic polytopes. In particular, we obtain generalizations of main theorems on difference dimension polynomials and their invariants to the case of weighted basic difference operators.

1. Introduction

Difference dimension polynomials, first introduced in [8], play the same role in the study of difference algebraic structures and systems of algebraic difference equations as is played by Hilbert polynomials in commutative algebra and algebraic geometry. Most of the known results on such polynomials (including algorithms for their computation) can be found in [7] and [9]. Furthermore, as it is shown in [9, Chapter 7], difference dimension polynomials have an important analytic interpretation: if a system of partial

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algebraic difference equations represents a system of equations in finite differences, then its difference dimension polynomial expresses the A. Einstein's strength of the system. This fact determines the importance of the study of difference dimension polynomials for the qualitative theory of algebraic difference equations.

Another important application of difference dimension polynomials is based on the fact that such a polynomial can be naturally assigned to any prime reflexive difference ideal of a finitely generated difference algebra. The relationship between prime reflexive difference ideals and their dimension polynomials has allowed one to obtain new interesting results on the Krull-type dimension of difference algebras and modules, as well as on local difference algebras (see, for example, [11], [9, Section 4.6] and [10]).

In this paper, we prove the existence and determine invariants of a dimension quasi-polynomial associated with a difference field extension with weighted basic translations. We also express a difference dimension quasi-polynomial as an alternating sum of Ehrhart quasi-polynomials associated with rational conic polytopes and show that, given a system of algebraic difference equations with weighted translations, one can assign to it a quasi-polynomial that can be viewed as an algebraic version of the Einstein's strength of the system. Note that systems of difference equations of these kind arise, in particular, from finite difference approximations of systems of PDEs with weighted derivatives, see, for example, [13] and [14]. One should also mention that the existence of Ehrhart-type dimension quasi-polynomials associated with weighted filtrations of differential and inversive difference modules was established by C. Dönch in his dissertation [4]. C. Dönch also proved the existence of dimension quasi-polynomials associated with finitely generated differential field extensions using the technique of weight Gröbner bases in the associated modules of Kähler differentials. This approach, however, cannot be applied in the case of non-inversive difference fields; the main tool used in our paper is the method of characteristic sets modified to the case of difference polynomials over a difference field with weighted translations.

2. Preliminaries

Throughout the paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of all non-negative integers, integers, rational numbers, and real numbers, respectively. If m is a positive integer, then by the product order on $\mathbb{N}^m$ we mean a partial order $\leq_P$ such that $(a_1, \dots, a_m) \leq_P (a'_1, \dots, a'_m)$ if and only if $a_i \leq a'_i$ for $i = 1, \dots, m$.

By a *difference ring* we mean a commutative ring R together with a finite set $\sigma = \{\alpha_1, \dots, \alpha_n\}$ of mutually commuting endomorphisms of R. The set σ is called a *basic set* of R and the endomorphisms α_i are called *translations*. We also say that R is a σ-ring. In what follows we assume that the translations are injective (this assumption is standard in most works on difference algebra). Furthermore, we will consider the free commutative semigroup of all power products $\tau = \alpha_1^{k_1} \dots \alpha_m^{k_m}$ $(k_1, \dots, k_m \in \mathbb{N})$ denoted by T (or T_σ). If $\theta, \tau \in T$ and there is $\tau' \in T$ such that $\tau = \theta\tau'$, we say that θ *divides* τ and write $\theta|\tau$. Otherwise, we write $\theta \nmid \tau$.