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Beta-hypergeometric probability distribution on symmetric matrices

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A B S T R A C T

In this paper, we first give some properties based on independence relations between matrix beta random variables of the first kind and of the second kind which are satisfied under a condition on the parameters of the distributions. We then show that with the matrix beta-hypergeometric distribution, the properties established for the beta distribution are satisfied without any condition on the parameters. The results involve many remarkable properties of the zonal polynomials with matrix arguments and the use of random matrix continued fractions. As a particular case, we get the results established for the real beta-hypergeometric distributions by Asci, Letac and Piccioni [1].

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1. Introduction

The real beta distributions of the first and of the second kind with parameters $p > 0$ and $q > 0$ are usually denoted by $\beta^{(1)}_{p,q}$ and $\beta^{(2)}_{p,q}$ respectively. They are given by

$$\beta^{(1)}_{p,q}(dx) = \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}\mathbf{1}_{(0,1)}(x)dx,$$

and

$$\beta^{(2)}_{p,q}(dx) = \frac{x^{p-1}(1+x)^{-(p+q)}}{B(p,q)}\mathbf{1}_{(0,+\infty)}(x)dx,$$

where $B(p,q)$ is the beta Euler function.

These distributions are among the most usual real distributions both in probability theory and in statistics, they are linked by many remarkable properties. It is in particular shown in [1] that

$$\text{if } W' \sim \beta^{(2)}_{a+a',a'} \text{ is independent of } X \sim \beta^{(1)}_{a,a'}, \text{ then } \frac{1}{1+W'X} \sim \beta^{(1)}_{a',a}, \tag{1.1}$$

and in [4] that

$$\text{if } W \sim \beta^{(2)}_{a+a',a},\ X \sim \beta^{(1)}_{a,a'},\ W' \sim \beta^{(2)}_{a+a',a'} \text{ are independent, then } \frac{1}{1+\frac{W}{1+W'X}} \sim X. \tag{1.2}$$

In these two properties, the random variables W and W' are beta distributed with first parameter equal to the sum of the parameters of the distribution of the variable X. Asci, Letac and Piccioni [1] have used the so-called real beta-hypergeometric distribution to extend these results to the general case where $W \sim \beta^{(2)}_{b,a}$ and $W' \sim \beta^{(2)}_{b,a'}$ with $b > 0$ not necessarily equal to $a + a'$. Recall that the real hypergeometric function ${}_pF_q$ is defined for positive numbers $a_1, ..., a_p; b_1, ..., b_q$, by

$${}_pF_q(a_1,...,a_p;b_1,...,b_q;x) = \sum_{n=0}^{\infty} \frac{(a_1)_n...(a_p)_n}{n!(b_1)_n...(b_q)_n}x^n, \text{ with } (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

The beta-hypergeometric distribution with parameters (a, a', b) is then defined by

$$\mu_{a,a',b}(dx) = C(a,a',b)x^{a-1}(1-x)^{b-1}\ {}_2F_1(a,b;a+a';x)\mathbf{1}_{(0,1)}(x)dx,$$

where

$$C(a,a',b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)_3F_2(a,a,b;a+b,a+a';1)}.$$

Note that the distribution $\mu_{a,a',b}$, reduces to a $\beta^{(1)}_{a,a'}$ when $b = a + a'$.