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Selected and satellite unknowns in linear differential systems

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A B S T R A C T

Consider the differential field $K = \overline{\mathbb{Q}}(x)$ with derivation d/dx. Let some unknowns (components of the unknown vector $y = (y_1, \dots, y_n)^T$) of a linear homogeneous differential system S over K be selected. Denote this set of selected unknowns by s. An unselected unknown y_j of the system S is called *satellite* for s if the minimal subfield of a Picard–Vessiot extension over K for S, that contains all selected components of all solutions to S, also contains y_j component of any solution. We present an algorithm for constructing the set of satellite unknowns for a given linear homogeneous differential system with selected unknowns.

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1. Introduction

Let $K = \overline{\mathbb{Q}}(x)$ be the differential field of rational functions with the usual derivation $' = \frac{d}{dx}$. Let S be a *normal* differential system, that is a differential system of the form

$$y' = Ay, \tag{1}$$

where $A \in \mathrm{M}_n(K)$ is the matrix of the system and $y = (y_1, \dots, y_n)^T$ is a vector of unknowns. Suppose that $s = \{y_{i_1}, \dots, y_{i_k}\}$ is a given nonempty set of *selected* unknowns (components of vector y) that does not contain all system unknowns (i.e. $0 < k < n$).

In this paper we introduce the notion of *satellite* unknowns. An unselected unknown y_j of a given system S is called *satellite* for the set of selected unknowns s if the j-th component of any solution to S belongs to a certain differential field extension F_s of K containing all selected components of any solution to S. The formal definition of a satellite unknown will be given in the next section.

Example 1. Consider the following differential system:

$$y' = \begin{bmatrix} 1/x & 0 & 0 \\ 3 & -1/x & 0 \\ 0 & 0 & 1 \end{bmatrix} y, \tag{2}$$

where $y = (y_1, y_2, y_3)^T$ is a vector of unknowns. The general solution to system (2) can be represented in the form

$$y = C_1 \begin{bmatrix} 1/x \\ 3 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ -1/x \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 0 \\ e^x \end{bmatrix}, \tag{3}$$

where C_1, C_2, C_3 are arbitrary constants. From (3) it follows that, for any solution to (2), its components y_1 and y_2 are rational functions in x. On the other hand, solution component y_3 cannot be represented in general as a rational function.

A set of satellite unknowns of the given system depends on a set of selected unknowns. If only the unknown y_1 of system (2) is selected (i.e. $s = \{y_1\}$), then it is clear that y_2 is a satellite unknown, but y_3 is not. Since $y_3 \in \overline{\mathbb{Q}}(x, e^x) \supset \overline{\mathbb{Q}}(x)$, we see that both y_1 and y_2 are satellite unknowns for the set of selected unknowns $s = \{y_3\}$.

In this paper we will also give an algorithm to find all satellite unknowns for a given linear differential system with a fixed set of selected unknowns. Contrarily to [7], we will give details of proposed algorithms and some illustrating examples.

The problem of satellite unknown testing can be considered in the context of the problem of solving linear differential systems with respect to the part of the unknowns. As it is noticed in the introduction of [1], computer algebra provides a lot of algorithms