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Bounds on the regularity of toric ideals of graphs

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ABSTRACT

Let G be a finite simple graph. We give a lower bound for the Castelnuovo–Mumford regularity of the toric ideal I_G associated to G in terms of the sizes and number of induced complete bipartite graphs in G. When G is a chordal bipartite graph, we find an upper bound for the regularity of I_G in terms of the size of the bipartition of G. We also give a new proof for the graded Betti numbers of the toric ideal associated to the complete bipartite graph $K_{2,n}$.

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1. Introduction

The last two decades have seen a significant dictionary developed between the algebraic invariants in the graded minimal free resolution of the edge ideal of a graph G and the graph-theoretic invariants of G (e.g., see [11,25]). Inspired by this work, we wish to work towards a similar dictionary between finite graphs and their associated toric ideals.

Given a finite simple graph $G = (V, E)$ with vertex set $V = \{x_1, \dots, x_n\}$ and edge set $E = \{e_1, \dots, e_r\}$, we abuse notation and define the polynomial rings $k[V] = k[x_1, \dots, x_n]$ and $k[E] = k[e_1, \dots, e_r]$ where k is any field. Define a monomial map $\pi : k[E] \to k[V]$ by $e_i \mapsto x_{i_1}x_{i_2}$ where $e_i = \{x_{i_1}, x_{i_2}\} \in E$. The kernel of $\pi : k[E] \to k[V]$, denoted I_G, is the *toric ideal defined by* G. It is well-known that the generators of I_G correspond to closed even walks in G, and in particular, I_G is a homogeneous ideal generated by binomials (see [25, Theorem 8.2.2], [17, Lemma 1.1], or [21] for a characterization of minimal generators). This construction is a specific instance of the more general construction of toric ideals; in the general case, the e_i's are mapped to distinct monomials in $K[V]$, and the toric ideal is the kernel of the corresponding map (see [20, Chapter IV] for more details).

Because I_G is a homogeneous ideal of $R = k[E]$, there is a *graded minimal free resolution* associated with I_G. That is, there exists a long exact sequence of the form

$$0 \to \bigoplus_{j\in\mathbb{N}} R(-j)^{\beta_{p,j}(I_G)} \to \bigoplus_{j\in\mathbb{N}} R(-j)^{\beta_{p-1,j}(I_G)} \to \cdots \to \bigoplus_{j\in\mathbb{N}} R(-j)^{\beta_{0,j}(I_G)} \to I_G \to 0$$

where $R(-j)$ is the graded R-module obtained by shifting the degrees of R by j and $p \le r$. The numbers $\beta_{i,j}(I_G)$ are the (i, j)-th *graded Betti numbers* of I_G.

Ideally, one would like to describe the $\beta_{i,j}(I_G)$'s in terms of combinatorial data of G. Some work in this direction has been carried out in [6]. In this paper, we focus on the *Castelnuovo–Mumford regularity* (or regularity) of I_G, that is,

$$\operatorname{reg}(I_G) = \max\{j - i \mid \beta_{i,j}(I_G) \neq 0\}.$$

Our project should be seen within the context of the much broader problem of understanding the regularity of an arbitrary toric ideal; e.g., see [5] for a method to compute the regularity of a toric ideal, and [23, Theorem 4.5] for an upper bound on the regularity of an arbitrary toric ideal. Motivation to study the regularity of toric ideals is also partially driven by the Eisenbud–Goto conjecture which states that the regularity of these ideals should be bounded in terms of the degree and codimension of the projective variety defined by the toric ideal (see [8,23]).

Our first main result is a lower bound on the regularity of I_G in terms of the presence of induced subgraphs that are isomorphic to complete bipartite graphs. Recall that the *complete bipartite graph* $K_{m,n}$ is the graph on the vertex set $\{x_1, \dots, x_m, y_1, \dots, y_n\}$ and edge set $E = \{\{x_i, y_j\} \mid 1 \le i \le m, \ \ 1 \le j \le n\}$. We show: