Advances in Applied Mathematics 86 (2017) 1–18

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

On the sign-imbalance of permutation tableaux

Joanna N. Chen [a,*], Robin D.P. Zhou [b]

[a] *College of Science, Tianjin University of Technology, Tianjin 300384, P.R. China*
[b] *College of Mathematics Physics and Information, Shaoxing University, Shaoxing 312000, P.R. China*

A R T I C L E I N F O

Article history:
Received 11 July 2016
Received in revised form 26 November 2016
Accepted 28 November 2016
Available online xxxx

MSC:
05A05
05A15

Keywords:
Permutation tableau
Sign-imbalance
Weak excedance
Bijection
Signed permutation
Symmetric permutation

A B S T R A C T

Permutation tableaux were introduced by Steingrímsson and Williams. Corteel and Kim defined the sign of a permutation tableau in terms of the number of unrestricted columns. The sign-imbalance of permutation tableaux of length n is the sum of signs over permutation tableaux of length n. They have obtained a formula for the sign-imbalance of permutation tableaux of length n by using generating functions and asked for a combinatorial proof. Moreover, they raised the question of finding a sign-imbalance formula for type B permutation tableaux introduced by Lam and Williams. We define a statistic $\overline{\mathrm{wm}}$ over permutations and show that the number of unrestricted columns over permutation tableaux of length n is equally distributed with $\overline{\mathrm{wm}}$ over permutations of length n. This leads to a combinatorial interpretation of the formula of Corteel and Kim. For type B permutation tableaux, we define the sign of a type B permutation tableau in term of the number of certain rows and columns. On the other hand, we construct a bijection between the type B permutation tableaux of length n and symmetric permutations of length $2n$ and we show that the statistic $\overline{\mathrm{wm}}$ over symmetric permutations of length $2n$ is equally distributed with the number of certain rows and columns over type B permutation tableaux of length n. Based on this correspondence and an

* Corresponding author.
E-mail addresses: joannachen@tjut.edu.cn (J.N. Chen), zhoudapao@mail.nankai.edu.cn (R.D.P. Zhou).

http://dx.doi.org/10.1016/j.aam.2016.11.012

involution on symmetric permutations of length $2n$, we obtain a sign-imbalance formula for type B permutation tableaux.

1. Introduction

This paper is concerned with two questions on the sign-imbalance of permutation tableaux of type A and type B, raised by Corteel and Kim [4]. Permutation tableaux were introduced by Steingrímsson and Williams [15]. They are related to the enumeration of totally positive Grassmannian cells [11,13,14,16], as well as a statistical physics model called Partially Asymmetric Exclusion Process (PASEP) [3,6–9]. For recent studies of permutation tableaux, see, for example, [1,2,4,5,12].

A permutation tableau is defined based on the Ferrers diagram of a partition λ for which zero parts are allowed. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition, that is, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$. The *Ferrers diagram of* λ is a left-justified arrangement with λ_i squares in the ith row. The *length* of a Ferrers diagram is the total number of rows and columns (including empty rows). In particular, the length of the Ferrers diagram of the empty partition is defined to be zero.

Given a Ferrers diagram F of length n, we label the rows and columns of F as follows. First, we give labels to the steps in the south-east border with $1, 2, \ldots, n$ from north-east to south-west. Then we label a row (resp. column) with i if the row (resp. column) contains the south (resp. west) step with label i. Notice that we may place a row label to the left of the first column and place a column label at the top of the first row, see Fig. 1.1. A row labeled with i is called *row* i and a column labeled with j is called *column* j. We use (i, j) to denote the cell in row i and column j.

For a partition λ, a *permutation tableau of shape* λ is a $0, 1$-filling of the Ferrers diagram of λ satisfying the following conditions:

1. Each column has at least one 1,
2. There is no 0 with a 1 above (in the same column) and a 1 to the left (in the same row).

The *length* of a permutation tableau is defined to be the length of the corresponding Ferrers diagram. Denote by $\mathcal{PT}(n)$ the set of permutation tableaux of length n. Fig. 1.2 illustrates a permutation tableau of length 12.

In their study of combinatorics of permutation tableaux in connection with PASEP, Corteel and Williams [6] introduced the concepts of a row-restricted 0 and an unrestricted row. A 0 in a permutation tableau is said to be *row-restricted* if there is a 1 above (in the same column). A row is called *unrestricted* if it does not contain any row-restricted 0. Otherwise, it is called a *restricted row*. Let T be a permutation tableau with k columns,