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Alternating sign matrices, extensions and related cones

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ABSTRACT

An *alternating sign matrix*, or ASM, is a $(0, \pm 1)$-matrix where the nonzero entries in each row and column alternate in sign, and where each row and column sum is 1. We study the convex cone generated by ASMs of order n, called the ASM cone, as well as several related cones and polytopes. Some decomposition results are shown, and we find a minimal Hilbert basis of the ASM cone. The notion of (± 1)-doubly stochastic matrices and a generalization of ASMs are introduced and various properties are shown. For instance, we give a new short proof of the linear characterization of the ASM polytope, in fact for a more general polytope. Finally, we investigate faces of the ASM polytope, in particular edges associated with permutation matrices.

1. Introduction

An *alternating sign matrix* ([20]), or ASM-matrix, is a $(0, \pm 1)$-matrix where the nonzero entries in each row and column alternate in sign, and where each row and column sum is 1. Let $\mathcal{P}_n$ be the set of $n \times n$ permutation matrices and let $\mathcal{A}_n$ be the set

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of $n \times n$ alternating sign matrices. The convex hull of $\mathcal{P}_n$ is the polytope Ω_n of doubly stochastic matrices with $\dim \Omega_n = (n-1)^2$ whose set of extreme points is $\mathcal{P}_n$ ([4,21]). Let Λ_n be the convex hull of $\mathcal{A}_n$. The next result was shown in [3,20], but we give a short argument for this fact.

Theorem 1. *The set of extreme points of Λ_n is $\mathcal{A}_n$.*

Proof. Let $x \in \{0, \pm 1\}^n$ be an alternating sign vector (a row or column in an ASM) which is a convex (positive) combination of other alternating sign vectors $x^{(k)}$s. In any position where x has a 1, each $x^{(k)}$ also has a 1, and the same holds for -1. Moreover, there must be a position j where x has 0 and some $x^{(k)}$ has 1, and therefore some $x^{(p)}$ has a -1. Choosing this position j closest possible to a position where x, and therefore $x^{(k)}$ and $x^{(p)}$, has a 1, we get a contradiction to the alternating property. This argument implies that no ASM can be a convex combination of ASMs different from it. □

It is also easy to see that $\dim \Lambda_n = (n-1)^2$. Note that we have $\mathcal{P}_n \subseteq \mathcal{A}_n$, and $\Omega_n \subseteq \Lambda_n$. It is an elementary fact that the maximum number of linearly independent permutation matrices in $\mathcal{P}_n$ equals $(n-1)^2 + 1$.

ASMs were defined by Mills, Robbins, and Ramsey who conjectured [16] that the number of $n \times n$ ASMs equals

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

This formula was proved in 1996 by Zeilberger [22], and independently by Kuperberg [14] who showed a bijection between ASMs and configurations of statistical physics known as square ice. An exposition of the fascinating origins of ASMs and a history of this conjecture including its connection with other combinatorial objects can be found in the book [5]. Spencer [19] showed that the above formula is asymptotic to

$$\left(\frac{3\sqrt{3}}{4}\right)^{n^2}.$$

Let X be a set in a vector space. We let Cone(X) denote the convex cone consisting of all nonnegative linear combinations of elements in X. A convex cone is called *rational* if it is generated by a set of rational vectors. X is called a *Hilbert basis* provided every integer vector in Cone(X) is expressible as a nonnegative *integer* linear combination of the vectors in X. It is proved in [13] that every pointed[1] rational cone is generated by a *unique* minimal Hilbert basis. We refer to this unique minimal Hilbert basis of a pointed rational cone Cone(X) as the *H-basis* of Cone(X). It is a classical result of

[1] A cone is pointed provided it contains no line.