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Critical ideals of signed graphs with twin vertices

Carlos A. Alfaro [a,1], Hugo Corrales [b,1], Carlos E. Valencia [c,*,1]

[a] *Banco de México, Calzada Legaria 691, módulo IV, Col. Irrigación, 11500 Ciudad de México, Mexico*

[b] *Escuela Superior de Economía, Plan de Agua Prieta No. 66, Col. Plutarco Elías Calles, 11340 Ciudad de México, Mexico*

[c] *Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del IPN, Apartado Postal 14-740, 07000 Ciudad de México, Mexico*

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This paper studies critical ideals of graphs with twin vertices, which are vertices with the same neighbors. A pair of such vertices are called replicated if they are adjacent, and duplicated, otherwise. Critical ideals of graphs having twin vertices have good properties and show regular patterns. Given a graph $G = (V, E)$ and $\mathbf{d} \in \mathbb{Z}^{|V|}$, let $G^{\mathbf{d}}$ be the graph obtained from G by duplicating $\mathbf{d}_v$ times or replicating $-\mathbf{d}_v$ times the vertex v when $\mathbf{d}_v > 0$ or $\mathbf{d}_v < 0$, respectively. Moreover, given $\delta \in \{0, 1, -1\}^{|V|}$, let

$$\mathcal{T}_\delta(G) = \{G^{\mathbf{d}} : \mathbf{d} \in \mathbb{Z}^{|V|} \text{ such that } \mathbf{d}_v = 0 \text{ if and only if } \delta_v = 0 \text{ and } \mathbf{d}_v\delta_v > 0 \text{ otherwise}\}$$

be the set of graphs sharing the same pattern of duplication or replication of vertices. More than one half of the critical ideals of a graph in $\mathcal{T}_\delta(G)$ can be determined by the critical ideals of G. The algebraic co-rank of a graph G is the maximum integer i such that the i-*th* critical ideal of G is trivial. We show that the algebraic co-rank of any graph in $\mathcal{T}_\delta(G)$ is equal to the algebraic co-rank of G^δ. Moreover, the algebraic co-rank can be determined by a simple evaluation of the critical ideals

* Corresponding author.
E-mail addresses: alfaromontufar@gmail.com (C.A. Alfaro), hhcorrales@gmail.com (H. Corrales), cvalencia@math.cinvestav.edu.mx, cvalencia75@gmail.com (C.E. Valencia).
URL: http://www.math.cinvestav.mx/cvalencia (C.E. Valencia).

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of G. For a large enough $\mathbf{d} \in \mathbb{Z}^{V(G)}$, we show that the critical ideals of $G^{\mathbf{d}}$ have similar behavior to the critical ideals of the disjoint union of G and some set $\{K_{n_v}\}_{\{v \in V(G) \mid \mathbf{d}_v < 0\}}$ of complete graphs and some set $\{T_{n_v}\}_{\{v \in V(G) \mid \mathbf{d}_v > 0\}}$ of trivial graphs. Additionally, we pose important conjectures on the distribution of the algebraic co-rank of the graphs with twins vertices. These conjectures imply that twin-free graphs have a large algebraic co-rank, meanwhile a graph having small algebraic co-rank has at least one pair of twin vertices.

1. Introduction and background

A signed multidigraph G_σ is a pair that consists of a multidigraph G (a digraph possibly with multiple arcs) and a function σ, called the sign, from the arcs of G into the set $\{1, -1\}$. Along the paper, all digraphs are allowed to have multiple signed arcs; when digraphs have neither multiple nor signed arcs, then we refer to them as graphs. Given a set of variables $X_G = \{x_u : u \in V(G)\}$ indexed by the vertices of G and a principal ideal domain (PID) $\mathcal{P}$, the generalized Laplacian matrix $L(G_\sigma, X_G)$ of G_σ is the matrix whose entries are given by

$$L(G_\sigma, X_G)_{uv} = \begin{cases} x_u & \text{if } u = v, \\ -\sigma(uv)m_{uv}1_{\mathcal{P}} & \text{otherwise,} \end{cases}$$

where m_{uv} is the number of arcs leaving u and entering v, and $1_{\mathcal{P}}$ is the identity of $\mathcal{P}$. Moreover, if $\mathcal{P}[X_G]$ is the polynomial ring over $\mathcal{P}$ in the variables X_G, then the critical ideals of G_σ are the determinantal ideals given by

$$I_i(G_\sigma, X_G) = \langle \{\det(m) : m \text{ is an } i \times i \text{ submatrix of } L(G_\sigma, X_G)\} \rangle \subseteq \mathcal{P}[X_G],$$

for all $1 \leq i \leq |V(G)|$. We say that a critical ideal is trivial when it is equal to $\langle 1_{\mathcal{P}} \rangle$. For simplicity, we write $I_i(G_\sigma, X)$ instead of $I_i(G_\sigma, X_G)$.

Definition 1.1. The algebraic co-rank $\gamma_{\mathcal{P}}(G_\sigma)$ of G_σ is the maximum integer i such that $I_i(G_\sigma, X)$ is trivial.

Since $I_n(G_\sigma, X) = \langle \det(L(G_\sigma, X)) \rangle \neq \langle 1 \rangle$, $\gamma_{\mathcal{P}}(G_\sigma) \leq n - 1$. The algebraic co-rank of a graph is closely related to combinatorial properties of the graph. For instance, if H_σ is an induced subgraph of G_σ, then $I_i(H_\sigma, X) \subseteq I_i(G_\sigma, X)$ for all $1 \leq i \leq |V(H)|$ (see [9, Proposition 3.3]). Therefore, $\gamma(H_\sigma) \leq \gamma(G_\sigma)$. Also, if $\alpha(G)$ and $\omega(G)$ denote the stability number and the clique number of G, respectively, then

$$\gamma_{\mathcal{P}}(G) \leq 2(n - \omega(G)) + 1 \text{ and } \gamma_{\mathcal{P}}(G) \leq 2(n - \alpha(G)),$$

see [9, Theorem 3.13].