

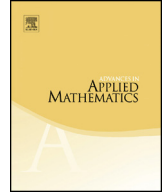


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## Critical ideals of signed graphs with twin vertices



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### ABSTRACT

This paper studies critical ideals of graphs with twin vertices, which are vertices with the same neighbors. A pair of such vertices are called replicated if they are adjacent, and duplicated, otherwise. Critical ideals of graphs having twin vertices have good properties and show regular patterns. Given a graph  $G = (V, E)$  and  $\mathbf{d} \in \mathbb{Z}^{|V|}$ , let  $G^{\mathbf{d}}$  be the graph obtained from  $G$  by duplicating  $\mathbf{d}_v$  times or replicating  $-\mathbf{d}_v$  times the vertex  $v$  when  $\mathbf{d}_v > 0$  or  $\mathbf{d}_v < 0$ , respectively. Moreover, given  $\delta \in \{0, 1, -1\}^{|V|}$ , let

$$\mathcal{T}_{\delta}(G) = \{G^{\mathbf{d}} : \mathbf{d} \in \mathbb{Z}^{|V|} \text{ such that } \mathbf{d}_v = 0$$

if and only if  $\delta_v = 0$  and  $\mathbf{d}_v \delta_v > 0$  otherwise}

be the set of graphs sharing the same pattern of duplication or replication of vertices. More than one half of the critical ideals of a graph in  $\mathcal{T}_{\delta}(G)$  can be determined by the critical ideals of  $G$ . The algebraic co-rank of a graph  $G$  is the maximum integer  $i$  such that the  $i$ -th critical ideal of  $G$  is trivial. We show that the algebraic co-rank of any graph in  $\mathcal{T}_{\delta}(G)$  is equal to the algebraic co-rank of  $G^{\delta}$ . Moreover, the algebraic co-rank can be determined by a simple evaluation of the critical ideals

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of  $G$ . For a large enough  $\mathbf{d} \in \mathbb{Z}^{V(G)}$ , we show that the critical ideals of  $G^{\mathbf{d}}$  have similar behavior to the critical ideals of the disjoint union of  $G$  and some set  $\{K_{n_v}\}_{\{v \in V(G) \mid \mathbf{d}_v < 0\}}$  of complete graphs and some set  $\{T_{n_v}\}_{\{v \in V(G) \mid \mathbf{d}_v > 0\}}$  of trivial graphs. Additionally, we pose important conjectures on the distribution of the algebraic co-rank of the graphs with twins vertices. These conjectures imply that twin-free graphs have a large algebraic co-rank, meanwhile a graph having small algebraic co-rank has at least one pair of twin vertices.

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### 1. Introduction and background

A signed multidigraph  $G_\sigma$  is a pair that consists of a multidigraph  $G$  (a digraph possibly with multiple arcs) and a function  $\sigma$ , called the sign, from the arcs of  $G$  into the set  $\{1, -1\}$ . Along the paper, all digraphs are allowed to have multiple signed arcs; when digraphs have neither multiple nor signed arcs, then we refer to them as graphs. Given a set of variables  $X_G = \{x_u : u \in V(G)\}$  indexed by the vertices of  $G$  and a principal ideal domain (PID)  $\mathcal{P}$ , the generalized Laplacian matrix  $L(G_\sigma, X_G)$  of  $G_\sigma$  is the matrix whose entries are given by

$$L(G_\sigma, X_G)_{uv} = \begin{cases} x_u & \text{if } u = v, \\ -\sigma(uv)m_{uv}1_{\mathcal{P}} & \text{otherwise,} \end{cases}$$

where  $m_{uv}$  is the number of arcs leaving  $u$  and entering  $v$ , and  $1_{\mathcal{P}}$  is the identity of  $\mathcal{P}$ . Moreover, if  $\mathcal{P}[X_G]$  is the polynomial ring over  $\mathcal{P}$  in the variables  $X_G$ , then the critical ideals of  $G_\sigma$  are the determinantal ideals given by

$$I_i(G_\sigma, X_G) = \langle \{\det(m) : m \text{ is an } i \times i \text{ submatrix of } L(G_\sigma, X_G)\} \rangle \subseteq \mathcal{P}[X_G],$$

for all  $1 \leq i \leq |V(G)|$ . We say that a critical ideal is trivial when it is equal to  $\langle 1_{\mathcal{P}} \rangle$ . For simplicity, we write  $I_i(G_\sigma, X)$  instead of  $I_i(G_\sigma, X_G)$ .

**Definition 1.1.** The algebraic co-rank  $\gamma_{\mathcal{P}}(G_\sigma)$  of  $G_\sigma$  is the maximum integer  $i$  such that  $I_i(G_\sigma, X)$  is trivial.

Since  $I_n(G_\sigma, X) = \langle \det(L(G_\sigma, X)) \rangle \neq \langle 1 \rangle$ ,  $\gamma_{\mathcal{P}}(G_\sigma) \leq n - 1$ . The algebraic co-rank of a graph is closely related to combinatorial properties of the graph. For instance, if  $H_\sigma$  is an induced subgraph of  $G_\sigma$ , then  $I_i(H_\sigma, X) \subseteq I_i(G_\sigma, X)$  for all  $1 \leq i \leq |V(H)|$  (see [9, Proposition 3.3]). Therefore,  $\gamma(H_\sigma) \leq \gamma(G_\sigma)$ . Also, if  $\alpha(G)$  and  $\omega(G)$  denote the stability number and the clique number of  $G$ , respectively, then

$$\gamma_{\mathcal{P}}(G) \leq 2(n - \omega(G)) + 1 \text{ and } \gamma_{\mathcal{P}}(G) \leq 2(n - \alpha(G)),$$

see [9, Theorem 3.13].

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