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Analysis of a mixed formulation of a bilateral obstacle problem

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ABSTRACT

In this paper we present a reformulation of a bilateral obstacle problem as a mixed formulation problem based on subdifferential of a continuous function of which the subdifferential can characterize the non-contact domain. Then we present the analysis of the discrete problem. We prove the convergence of the approximate solution to the exact one and we provide an error estimate. This formulation was established in an abstract way, then the theoretical results was applied to a bilateral obstacle problem.

1. Introduction

Let Ω be a bounded and open domain in $\mathbb{R}^n$ with smooth boundary, and let ψ_1, ψ_2 and η be elements of $H^1(\Omega)$ such that $\psi_1 \leq \psi_2$ in Ω and $\psi_1 \leq \eta \leq \psi_2$ on $\partial\Omega$. We consider the following variational inequality:

$$\begin{cases} \text{Find } w \in K_\eta := \left\{ v \in H^1(\Omega) \mid \psi_1 \leq v \leq \psi_2 \text{ a.e. in } \Omega \text{ and } v = \eta \text{ on } \partial\Omega \right\} \text{ such that} \\ \displaystyle\int_\Omega \nabla w \cdot \nabla (v - w) dx + \int_\Omega f(v - w) dx \geq 0 \quad \forall v \in K_\eta, \end{cases} \tag{1}$$

where f is an element of $L^2(\Omega)$. This problem is called the bilateral obstacle problem. It is well known that problem (1) admits a unique solution w, and, if $\Delta\psi_1, \Delta\psi_2 \in L^2(\Omega)$, then w is an element of $H^2(\Omega)$, see [7,8,11,14]. There are several alternative formulations of the obstacle problem, see for instance, [2,3,8,10–12].

The present work falls into the evolution of the approach proposed in [12] to reformulate the bilateral obstacle problem as a set value equation problem, and developed in an abstract framework in [2]. This abstract formulation can be applied to problem (1) for the special case where w, ψ_1, ψ_2 vanishes on $\partial\Omega$. In this paper we generalize this abstract formulation to be applied to the bilateral obstacle problem in the general case where ψ_1, ψ_2 and $\eta \in H^1(\Omega)$.

We introduce a new mixed formulation of the abstract bilateral problem based an appropriate variational inequality of the second kind and the notion of the subdifferential of a convex continuous function. This formulation is equivalent to a saddle point problem of which the Lagrange multipliers characterize the non-contact domain of the solution with the obstacles ψ_1 and ψ_2. Next we approximate this problem by a sequence of discrete ones. Then we provide an error estimate and prove the convergence of the approximate solution to the exact one. Finally we apply the theoretical results to the bilateral obstacle problem (1). Note that such theoretical work can be applied to other similar problems.

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The paper is structured as follows. In Section 2, we present the abstract problem. In Section 3, we reformulate the problem as a mixed formulation problem. In Section 4, we present the discrete problem. In Section 5, we present the analysis of the discrete problem. In Section 6, we give an application to the bilateral obstacle problem.

2. Abstract problem

In [12] the authors established a direct reformulation of the bilateral obstacle problem defined on functional spaces, while in [2] the authors proposed an abstract formulation applicable to this problem in the particular case where, ψ_1 and $\psi_2 \in H_0^1(\Omega)$ and $f \in L^2(\Omega)$, so this formulation needs two Hilbert spaces H and V. In this section, we are interested in developing an abstract framework to study the bilateral obstacle problem in the general case where the obstacles do not necessarily vanishes on $\partial\Omega$. This leads us to consider an intermediate space M.

Let H be a Hilbert space with the inner product $(\cdot,\ \cdot)$, and let M be a subspace of H equipped with a norm denoted by $\|\cdot\|_M$. Let Y be a convex and closed cone in M. We set

$$V := Y \cap (-Y).$$

Then, it is easy to see that V is a subspace of M. We assume that $a(\cdot,\ \cdot)$ is a continuous bilinear form on M and defines an inner product on V, such that we have

$$V \subset M \tag{2}$$

with continuous injection, and

$$V \subset H \subset V' \tag{3}$$

with continuous and dense injections, where V' denotes the dual space of V. Let X be a convex and closed cone in H. Then, every element $v \in H$ can be written in the following way [13]:

$$v = v^+ + v^- \quad \text{with} \quad (v^+, v^-) = 0,$$

where v^+ and v^- are, respectively, the projections of v onto X and onto its polar cone defined by

$$X^\circ = \{v \in H : (v, w) \leq 0 \quad \forall w \in X\}.$$

We assume the following hypotheses:

$$X = -X^\circ. \tag{4}$$

$$\forall v \in Y : v^- \in V. \tag{5}$$

$$\forall v \in Y : a(v^+, v^-) \geq 0. \tag{6}$$

$$\forall v, u \in Y : v^+ + u^+ - (v+u)^+ \in X. \tag{7}$$

$$\forall v \in Y;\ \forall u \in (-Y) : v^+ + u^+ - (v+u)^+ \in X. \tag{8}$$

For example, let Ω be a bounded and open domain in $\mathbb{R}^n$ with smooth boundary. If we take:

- $H := L^2(\Omega)$, with $(u, v) := \int_\Omega uv dx \quad \forall u, v \in L^2(\Omega)$,
- $M := H^1(\Omega)$, with $a(u, v) := \int_\Omega \nabla u . \nabla v dx \quad \forall u, v \in H^1(\Omega)$,
- $X := \{v \in L^2(\Omega) :\ v \geq 0 \text{ a.e. in } \Omega\}$,
- $Y := \{v \in H^1(\Omega) :\ v \geq 0 \text{ on } \partial\Omega\}$,
- $V := H_0^1(\Omega)$,

we can see that the foregoing assumptions are verified.
Hypotheses (5)–(8) imply the following results which will be needed in the sequel of this paper.

Lemma 2.1.

(a) $\forall v \in Y$: *we have* $v^+ \in Y$ *which makes sense to the hypothesis* (6).
(b) $\forall v \in V$: *we have* $v^+, v^- \in V$.
(c) $\forall v \in V$, $\forall u \in Y$: *we have* $(v+u)^+ - u \in V$.
(d) $\forall v, u \in Y$: *we have* $v^+ + u^+ - (v+u)^+ \in V$.
(e) $\forall v \in (-Y)$, $\forall u \in Y$ *such that* $u - v \in X$ *we have* $(v - u^-)^- = v^- - u^-$.