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# On the application of the method of fundamental solutions to boundary value problems with jump discontinuities

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ABSTRACT

Two meshfree methods are proposed for the numerical solution of boundary value problems (BVPs) for the Laplace equation, coupled with boundary conditions with jump discontinuities. In the first case, the BVP is solved in two steps, using a subtraction of singularity approach. Here, the singular subproblem is solved analytically while the classical method of fundamental solutions (MFS) is applied for the solution of the regular subproblem. In the second case, the total BVP is solved using a variant of the MFS where its approximation basis is enriched with a set of harmonic functions with singular traces on the boundary of the domain. The same singularity-capturing functions, motivated by the boundary element method (BEM), are used for the singular part of the solution in the first method and for augmenting the MFS basis in the second method. Comparative numerical results are presented for 2D problems with discontinuous Dirichlet boundary conditions. In particular, the inappropriate oscillatory behavior of the classical MFS solution, due to the Gibbs phenomenon, is shown to vanish.



## 1. Introduction

The method of fundamental solutions (MFS) is a Trefftz type meshfree numerical method which can be applied for the approximate solution of certain boundary value problems (BVPs), usually with continuous boundary data, e.g. [1–4]. For smooth domains, spectral convergence has been proven in [5–7], in the 2D (or complex) case, assuming that the boundary condition is an analytic (entire) function.

However, when the boundary of the domain has singularities, e.g. corners, cracks, etc., or when the boundary conditions are not sufficiently regular, the analytic MFS shape functions prove to be inappropriate for fitting a possibly singular solution. While accuracy problems due to geometric singularities have been successfully addressed, e.g. [8,9], the lack of convergence of the MFS in the case of BVPs with singular boundary conditions has been somehow neglected in the MFS literature.

Most commonly, the Motz benchmark problem [10] is considered when newly developed meshfree methods are tested against BVPs with singular boundary conditions, e.g. [11,12]. In the Motz problem an abrupt change in the type of boundary condition occurs, from Dirichlet to Neumann, leading to discontinuities of the partial derivatives of the exact solution on the boundary. However, the solution of the BVP is globally continuous in the closure of the domain and this problem may

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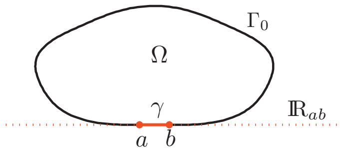


**Fig. 1.** Model problem settings in 2D.

be solved by adding particular solutions of the PDE, derived in polar coordinates, to the MFS approximation basis, as it has been done for the 2D Helmholtz BVP in [9].

In this work, we consider Dirichlet boundary conditions that may present a jump discontinuity in the function itself. Therefore, the exact solution of the BVP has a singular trace on the boundary which is a discontinuous function in the tangential direction. It is known that in this case the approximate solution by the MFS suffers from the Gibbs phenomenon, e.g. [13]. Also, in general, the type of singularity exhibited by the fundamental solution is different from the one shown by the exact solution and therefore no accurate numerical results may be achieved by the classical MFS, even if source points are taken arbitrarily close to the boundary. On the other hand, continuous particular solutions similar to the ones derived in [9] are also not appropriate for fitting the discontinuous Dirichlet boundary function.

In order to correctly recreate the discontinuity of the exact solution on the boundary we will include a set of exact solutions with discontinuous boundary traces in the MFS approximation basis. The definition of such special functions is motivated by the formulation of the classical boundary element method (BEM) and they can be derived analytically, by evaluating a double layer potential on a boundary segment.

This paper extends previous works by Alves and Leitão [8] and Antunes and Valtchev [9]. Here, the coupling with BEM is not in the sense of obtaining two different solutions by implementing two different methods (see for instance Tadeu et al. [14]), it is in the sense of adding a small number of BEM-type singularity-capturing basis functions to the MFS approximation space, while preserving all meshfree characteristics of the original method.

The technique presented here for the Laplace equation can be extended to other partial differential equations, with a known fundamental solution, provided the evaluation of the double layer potential on a boundary segment can be carried out analytically or numerically.

The paper is divided as follows. In Section 2 we briefly describe the subtraction of singularity approach and split the BVP into a regular and a discontinuous subproblem. In Section 3 we focus on the Dirichlet problem for the 2D Laplace equation, posed in a square domain and derive the special harmonic functions that will be used for the analytic solution of the singular subproblem. In Section 4, a modification of the classical MFS is presented, where the special harmonic functions derived in the previous section are appended to the approximation basis. Finally, in Section 5 we present numerical results that illustrate the performance of the two approaches.

## 2. Subtraction of singularity approach

Let $\Phi$ be a fundamental solution of the elliptic differential operator $\mathcal{L}$, with constant coefficients, thus satisfying, in the sense of distributions

$$\mathcal{L}\Phi = \delta,$$

where $\delta$ is the Dirac distribution.

It is well known, e.g. [1–3], that the method of fundamental solutions can be applied for the Dirichlet boundary value problem

$$(\mathcal{P})\begin{cases}\mathcal{L}u = 0 & \text{in } \Omega\\ u = \mathcal{G} & \text{on } \Gamma := \partial\Omega,\end{cases}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3, \ldots)$ is a bounded simply connected domain, but its accuracy is greatly affected if the boundary data $\mathcal{G}$ is not represented by a sufficiently regular function. In particular, if the function $\mathcal{G}$ is discontinuous, the approximate solution presents a typical oscillatory behavior known as the Gibbs phenomenon, similar to the one observed for Fourier approximations of discontinuous functions, e.g. [13].

In this paper we will address only the 2D case, and therefore $\Gamma$ will be a closed curve, see Fig. 1. Also, in order to simplify the presentation, we will assume that $\mathcal{G}$ presents at most two discontinuities, located at the end points $a$ and $b$ of the straight line segment

$$\gamma := ]a, b[ = \{(1-t)a + t\,b : t \in ]0, 1[\} \subset \Gamma.$$

The function $\mathcal{G}$ can then be rewritten as:

$$\mathcal{G}(x) = \begin{cases} G(x) & \text{on } \Gamma_0 = \Gamma \backslash \bar{\gamma}\\ g(x) & \text{on } \gamma,\end{cases}$$

where $G$ and $g$ are regular functions, for example $G \in C^m(\Gamma_0)$ and $g \in C^m(\gamma)$, and at least one of the following conditions is true: $G(a) \neq g(a)$ or $G(b) \neq g(b)$.