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Asymptotics for varying discrete Sobolev orthogonal polynomials☆

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ABSTRACT

We consider a varying discrete Sobolev inner product such as

$$(f,g)_S = \int f(x)g(x)d\mu + M_n f^{(j)}(c)g^{(j)}(c),$$

where μ is a finite positive Borel measure supported on an infinite subset of the real line, c is adequately located on the real axis, $j \geq 0$, and $\{M_n\}_{n\geq 0}$ is a sequence of nonnegative real numbers satisfying a very general condition. Our aim is to study asymptotic properties of the sequence of orthonormal polynomials with respect to this Sobolev inner product. In this way, we focus our attention on Mehler-Heine type formulae as they describe in detail the asymptotic behavior of these polynomials around c, just the point where we have located the perturbation of the standard inner product. Moreover, we pay attention to the asymptotic behavior of the (scaled) zeros of these varying Sobolev polynomials and some numerical experiments are shown. Finally, we provide other asymptotic results which strengthen the idea that Mehler–Heine asymptotics describe in a precise way the differences between Sobolev orthogonal polynomials and standard ones.

1. Introduction

Orthogonal polynomials with respect to the varying inner product

$$(f,g)_S = \int f(x)g(x)d\mu + M_n f^{(j)}(c)g^{(j)}(c), \quad j \geq 0, \tag{1}$$

where c is adequately located on the real axis, have been considered in some papers (see [6] and [7] and the references therein) recently. In these papers the authors focus their attention on Mehler–Heine asymptotics given the relevance of this type of asymptotics for describing the differences between the sequences of orthogonal polynomials with respect to (1) and

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those with respect to μ. The main goal of this paper is to give a final and global vision of the Mehler–Heine asymptotics for the orthogonal polynomials with respect to (1). In fact, whether μ has bounded or unbounded support will not be relevant to the results that we will provide. Therefore, all the previous results about this type of asymptotics for varying Sobolev orthogonal polynomials in the aforementioned papers are particular cases of Theorem 1 (or of its symmetric version).

Mehler–Heine asymptotics were introduced for Legendre polynomials by H. E. Heine and G. F. Mehler in the 19th century. In Szegő's book [16, Sec. 8.1] we can find the corresponding Mehler–Heine formulae for classical continuous orthogonal polynomials: Jacobi, Laguerre, and Hermite. As a consequence, using Hurwitz's theorem, the asymptotic behavior of the scaled zeros of these families of polynomials is deduced. Along this century, several authors have paid attention to this type of asymptotics in different contexts such as multiple orthogonal polynomials [18], Sobolev orthogonal polynomials (partially cited in the surveys [8] and [10]), generalized Freud polynomials [3], exceptional orthogonal polynomials [5], among others.

Regarding the varying Sobolev inner product (1), Mehler–Heine asymptotics of the corresponding orthogonal polynomials have been studied for measures μ related to the Jacobi and Laguerre weight functions in [6] and [7]. In both papers the techniques used involve particular properties of Jacobi/Laguerre orthogonal polynomials. However, in [15] a general approach is given for the non–varying case. With that new technique the authors obtain asymptotic results for general measures which can have either bounded or unbounded support. Therefore, our aim in this paper is to apply this method to the varying case. In the present paper, we conclude the study of the asymptotics for these varying orthogonal polynomials.

Thus, we consider the inner product (1) where $\{M_n\}_{n\geq 0}$ is a sequence of nonnegative real numbers satisfying the following general condition

$$\lim_{n\to\infty} M_n K_{n-1}^{(j,j)}(c,c) = L \in [0,+\infty], \tag{2}$$

where $K_{n-1}^{(j,k)}(x,y)$ denotes the partial derivatives of the nth kernel for the sequence of polynomials $\{p_n\}_{n\geq 0}$ orthonormal with respect to the finite positive Borel measure μ, i.e.

$$K_n^{(j,k)}(x,y) = \frac{\partial^{j+k}}{\partial x^j \partial y^k} K_n(x,y) = \sum_{i=0}^{n} p_i^{(j)}(x) p_i^{(k)}(y), \qquad j,k \in \mathbb{N}\cup\{0\}.$$

Notice that condition (2) is even more general than the condition stated for $\{M_n\}_{n\geq 0}$ in [6] and [7]. In fact, we can observe $M_n K_{n-1}^{(j,j)}(c,c)$ is nonnegative for each n (actually it is positive for almost every n). Thus, condition (2) is very general since we admit that this sequence can be either convergent or divergent ($L=+\infty$).

We are going to work with orthonormal polynomials, so we denote by $\{q_n\}_{n\geq 0}$ the sequence of orthonormal polynomials with respect to the inner product (1). In fact, for each n, we have a square tableau of orthonormal polynomials $\{q_k^{(M_n)}\}_{k\geq 0}$ but we deal with the diagonal of this tableau $\{q_n^{(M_n)}\}_{n\geq 0} =: \{q_n\}_{n\geq 0}$. Along the paper we will use the notation $f_n \simeq g_n$ to indicate that $\lim_{n\to\infty} f_n/g_n = 1$.

We will prove that for the varying case we obtain three different Mehler–Heine formulae depending on the value of L, or equivalently, on the size of the sequence $\{M_n\}_{n\geq 0}$. That is relevant since, on one hand, we will show that the term $M_n f^{(j)}(c) g^{(j)}(c)$ influences on the local asymptotic behavior of q_n and, on the other hand, it is limited by the size of L (for example, when $L=0$ there is no influence, and the case $L=+\infty$ includes, as a very particular situation, the *constant case* $M_n = M > 0$, for all n).

The structure of the paper is the following. In Section 2 we establish the necessary background about the sequence of varying orthonormal polynomials q_n. In Section 3 we provide the Mehler–Heine asymptotics for the sequence $\{q_n\}_{n\geq 0}$ distinguishing two cases: either μ is symmetric or μ is nonsymmetric. In Section 4 the consequences of the Mehler–Heine formulae on the asymptotic behavior of the zeros of q_n are shown. In Section 5, we obtain the outer relative asymptotics between the families of polynomials $\{q_n\}_{n\geq 0}$ and $\{p_n\}_{n\geq 0}$ when μ is a measure that belongs to Szegő's class, as well as the Plancherel–Rotach asymptotics when μ has an unbounded support. Finally, we illustrate the asymptotic behavior of the zeros given in Section 4 with an example involving the Hermite weight.

2. Varying discrete Sobolev orthonormal polynomials

Let $\{p_n\}_{n\geq 0}$ ($p_n(x) = \gamma_n x^n +$ lower degree terms, and $\gamma_n > 0$) be the sequence of orthonormal polynomials with respect to the measure μ and $\{q_n\}_{n\geq 0}$ ($q_n(x) = \tilde{\gamma}_n x^n +$ lower degree terms, and $\tilde{\gamma}_n > 0$) the sequence of orthonormal polynomials with respect to the inner product (1). In addition, we denote by $\{p_n^{[2i]}\}_{n\geq 0}$ the sequence of orthonormal polynomials with respect to the measure $d\mu_{2i}(x) = (x-c)^{2i} d\mu(x)$, $i \geq 0$ and $c \in \mathbb{R}\backslash \mathrm{supp}(\mu)$. We notice that the leading coefficient of all the orthonormal polynomials considered in this paper are taken positive.

A useful connection formula between the families of polynomials $\{q_n\}_{n\geq 0}$ and $\{p_n^{[2i]}\}_{n\geq 0}$ was given in [15, Th. 1] for non–varying discrete Sobolev orthogonal polynomials. The proof for the varying case is totally analogous, therefore we omit it. Thus, we have