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A meshless symplectic algorithm for nonlinear wave equation using highly accurate RBFs quasi-interpolation☆

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ABSTRACT

This study suggests a high-order meshless symplecitc algorithm for Hamiltonian wave equation by using highly accurate radial basis functions (RBFs) quasi-interpolation operator. The method does not require solving a resultant full matrix and possesses a high order accuracy compared with existing numerical methods. We also present a theoretical framework to show the conservativeness and convergence of the proposed symplectic method. As the numerical experiments shown, it not only offers a high order accuracy but also has a good property of long-time tracking capability.

1. Introduction

The nonlinear wave equation (NLW)

$$u_{tt} - u_{xx} + G'(u) = 0 \tag{1.1}$$

where $G'(u) : \mathbb{R} \mapsto \mathbb{R}$ is a smooth function, is usually considered as an illustration of nonlinear phenomena, such as the propagation of dislocation in crystal, the behavior of elementary particles [8], etc. It is a classical example of the infinite-dimensional Hamiltonian systems [20]. Hamiltonian PDEs preserve the symplectic structure. To compute an accurate solution for these Hamiltonian systems, one hopes that the numerical solution will hold this property too. Until recently, a lot of numerical procedures for Hamiltonian PDEs have been developed. For example, the symplectic finite difference method (SFDM) by [2,3,9,10]; the symplecitc Fourier pseudospectral method by McLachlan [15]; the symplectic finite element method (SFEM) by Zhen et al. [28], etc. However, most of those methods are depend on mesh generation, which is hard for problems with very complicated and irregular geometries.

More recently, a meshless symplectic algorithm for multi-variate Hamiltonian PDEs with radial basis functions interpolation is proposed by Wu and Zhang [26]. The method involves two steps. First, discretization in space by using RBFs interpolation transforms the PDE into a finite-dimensional Hamiltonian system (ODEs). In the second step, the resulting system of ODEs is integrated by using symplectic integrators in time. It is a meshless computational algorithm which does not require the generation of a grid as in SFDM or a mesh as in the SFEM. It is easy to implement with the scattered knots, offers a high order of convergence and possesses a long-time tracking capability for solving Hamiltonian PDEs. However,

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the proposed method faces a serious ill-conditioning problem due to the use of the RBFs as a global interpolation, and the results are sensitive to a shape parameter c [16]. For more information about the RBFs interpolation (collocation) method, we refer readers to the latest literatures [5–7,18,19,21,22], etc. To avoid solving a large scaled linear system of equations, RBFs quasi-interpolation method thereby has caught the attentions of many researchers.

In the literature [25], the symplectic algorithm, which is constructed by using first degree multiquadric (MQ) quasi-interpolation, possesses a lower convergence order. Therefore we want to generalize it by using highly accurate MQ quasi-interpolations. In this paper, with the help of cubic MQ quasi-interpolation, a high-order meshless symplectic algorithm for the nonlinear wave equation will be developed. It is worth pointing out that the multivariate problems can be settled by using tensor product approach for grided data or the method showed in [23] for scattered data, we thereby confined ourselves to the problem of one real variable.

The layout of the paper is as follows. In Section 2, preliminaries about cubic multiquadric quasi-interpolation are calculated and the properties about cubic multiquadric function are prepared. In Section 3, a symplectic cubic multiquadric quasi-interpolation method for solving nonlinear wave equation is introduced. Both the conservativeness and convergence of the proposed symplectic method are investigated in Section 4. In Section 5, three examples are given to verify the effectiveness of the method. The last section is dedicated to a brief conclusion.

2. Cubic MQ quasi-interpolation and cubic MQ function

2.1. Cubic MQ quasi-interpolation

It is well known the multiquadric is one of the most commonly used kernel functions in meshless methods. Multiquadric kernels were proposed by Hardy [12]. Franke tested a lot of numerical experiments, among of which MQ functions performed best [11]. Bestson and Dyn gave the exact definition of MQ-B-Splines for the first time and described its properties [1]. Base on the definition of MQ-B-Splines, Zhang and Wu developed a cubic MQ quasi-interpolation collocating with non-uniformly distributed data [27]. Indeed, the development of RBFs quasi-interpolation method as a meshfree numerical algorithm, has drawn the attention of many researchers in science and engineering. Even [13] applied the cubic MQ quasi-interpolation to the field of financial engineering. More details about MQ quasi-interpolation can be found in [1,14,24,27], etc.

Given a points sequence

$$\mathbf{x} = \cdots < x_{j-1} < x_j < x_{j+1} < \cdots, \qquad h := \max_j (x_j - x_{j-1})$$

with $x_{\pm j} \to \pm\infty$ when $j \to \pm\infty$. A cubic multiquadric quasi-interpolation of a function $f : \mathbb{R} \mapsto \mathbb{R}$ on $\mathbf{x}$ is defined as

$$(\mathcal{L}f) = \sum f(x_j)\Psi_j(x), \tag{2.1}$$

where

$$\Psi_j(x) = \Psi(x - x_j) = \frac{\psi_{j+1}(x) - \psi_j(x)}{2(x_{j+2} - x_{j-1})} - \frac{\psi_j(x) - \psi_{j-1}(x)}{2(x_{j+1} - x_{j-2})}$$

and $\psi_j(x)$ are the following linear combinations of cubic MQ function, i.e.

$$\psi_j(x) = \psi(x - x_j) = \frac{(\phi_{j+1} - \phi_j)/(x_{j+1} - x_j) - (\phi_j - \phi_{j-1})/(x_j - x_{j-1})}{x_{j+1} - x_{j-1}},$$

with $\phi(x) = \sqrt{(x^2 + c^2)^3}$ and $\phi_j = \phi(x - x_j)$, where c is a shape parameter. As discussed in [27], the above quasi-interpolation possesses shape-preserving and high-order approximation properties. Besides, the function Ψ_j satisfies

$$\sum_j \Psi_j(x) = 1, \tag{2.2}$$

and the value $\Psi_{ij} = \Psi_j(x_i)$ is monotonically approaching zero when the distance between any two points x_i and x_j increase. Given any $\epsilon > 0$, one can find a positive integer M so that the summation (2.2) becomes

$$1 - \sum_{j=i-M}^{j=i+M} \Psi_j(x) < \epsilon, \tag{2.3}$$

for any fixed $x = x_i$. This implies that the value of Ψ_{ij} is close to zero if $|i - j| > M$. Meanwhile, its first- and second-order derivatives satisfy

$$\sum_{j=i-M}^{j=i+M} \frac{\partial \Psi_j(x_i)}{\partial x} < \epsilon, \quad \sum_{j=i-M}^{j=i+M} \frac{\partial^2 \Psi_j(x_i)}{\partial x^2} < \epsilon, \; i \in \mathbb{Z}. \tag{2.4}$$

That means the matrix $\boldsymbol{\Psi} = \{\Psi_j(x_i)\}$ and $\boldsymbol{\Psi}_2 = \{\frac{\partial^2 \Psi_j(x_i)}{\partial x^2}\}$ can then be treated approximately as bounded matrices with bandwidth of $2M + 1$, which is computationally efficient in numerical procedure.