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Hybrid difference scheme for singularly perturbed reaction-convection-diffusion problem with boundary and interior layers☆

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ABSTRACT

A singularly perturbed second order ordinary differential equation having two small parameters with a discontinuous source term is considered. The presence of two parameters gives rise to boundary layers of different widths and the discontinuous source term generates interior layers on both sides of the discontinuous point. Theoretical bounds are derived. The problem is solved numerically with finite difference methods on a Shishkin mesh. The discretization combines a five point second order scheme at the interior layer together with the standard central, mid-point and upwind difference scheme for other regions. This combination is used in order to obtain almost second order convergence for the considered problem. Parameter uniform error bounds for the numerical approximation are established. Numerical results are presented to illustrate the convergence of the numerical approximations.

1. Introduction

Singularly perturbed differential equations arise in many areas of applied mathematics such as transport phenomena in chemistry and biology [1], chemical reactor theory [2–4], lubrication theory [5], elasticity [6], theory of plates and shells [7], fluid dynamics [8], D-C motors [9], etc. Certain types of problem arise in the models of chemical reactors and the governing equations, which determine chemical species concentrations and fluid temperature, based on conservation laws involving chemical reactions, diffusion, advection and external sources [10]. The differential equation depends on a small positive parameter (ε) multiplying the highest derivative term. When the parameter tends to zero ($\varepsilon \to 0$) the problem has a limiting solution, which is the solution of the reduced problem [11] and the regions of non-uniform convergence lie near the boundary, known as boundary layers. These problems have steep gradients in the narrow layer regions of the domain in consideration. This causes severe hurdles in the computations for classical numerical methods. In order to capture the layers, a large number of special purpose methods have been developed by the researchers to provide accurate numerical solutions, which cover second order equations with single parameter for smooth [11–13] and non-smooth data [14–17].

Two parameter Singularly Perturbed Problems (SPPs) are not considered in broad area and robust numerical methods for solving two parameter problem are not much identified when compared to one parameter SPPs. In two parameter

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(reaction-convection-diffusion) problem both the diffusion and convection terms are multiplied by small parameters. Depending on the size of the parameters the solution of the problem may absorb exponential layers at the boundary points of the domain. Several authors have mapped convection-diffusion problem with smooth data. For instance, Shishkin and Titov [18] applied an exponential fitted difference scheme on an equidistribution mesh. O'Riordan et al. [19] proved a first order parameter uniform method with standard upwind finite difference operator. Linß [20] has suggested a streamline diffusion finite element method for reaction-diffusion-convection type problem. A robust a *posteriori* error estimate in the maximum norm is derived. An adaptive moving mesh method for linear 1-D parabolic reaction-convection-diffusion initial-boundary value problems with two small parameters is examined [21]. Higher order finite difference scheme on a piecewise uniform mesh of Shishkin and Bakhavalov type is constructed for solving quasi-linear boundary value problems with small parameters [4].

Gracia et al. [22] have discussed a parameter uniform second order method on Shishkin mesh. A novel fitted operator method is constructed by Patidar [23] to obtain a parameter uniform first order convergence. But not many results and analysis are known about parameter uniform numerical methods to solve two parameter problems with non-smooth data. The non-smoothness would cause interior layers with different scales. Analysis and robust numerical methods to this type of problem are very challenging. Shanthi et al. [24] considered a simple fitted mesh method to solve reaction-convection-diffusion problem with non-smoothness occurring on the source term and derived first order convergence. For a similar type of problem the authors have used a hybrid difference scheme with average technique on an adaptive mesh to improve the order of convergence [25]. Bhakvalov and Shishkin meshes are studied by Mohapatra [26] for the convection-reaction-diffusion problem with discontinuity in the convection coefficient. But the results are not improved. Clavero et al. [27] have considered a one-dimensional two parameter singularly perturbed parabolic problem with the source term having discontinuity of first kind on the degeneration line. This type of problems are found in simple models of diffraction. The authors also have presented some collected works on SPPs of PDE with discontinuous data and degenerating convective terms. The problem which we have considered falls under the nature of this type.

In this paper, we are interested to improve the order of accuracy for the more general form of linear one-dimensional second order SPP with a discontinuity in source term. With the motivation from Gracia et al. [22] and Shanthi et al. [24], we consider a singularly perturbed reaction-convection-diffusion problem in one dimension with a discontinuous source term of the form

$$Lu(x) \equiv \varepsilon u''(x) + \mu a(x)u'(x) - b(x)u(x) = f(x), \quad \forall\, x \in (\Omega^- \cup \Omega^+), \tag{1}$$

$$u(0) = u_0, \quad u(1) = u_1. \tag{2}$$

The notation $\overline{\Omega} = [0,1]$, $\Omega^- = (0,d)$ and $\Omega^+ = (d,1)$ are introduced for convenience. The coefficients $a(x)$ and $b(x)$ are sufficiently smooth functions in $\overline{\Omega}$ and $f(x)$ is sufficiently smooth in $(\Omega^- \cup \Omega^+) \cup \{0,1\}$. Also, $f(x)$ and its derivatives have a discontinuity at $d \in \Omega = (0,1)$, $|[f](d)| \leq C$, $0 < \varepsilon < <1$, $0 \leq \mu \leq 1$, $a(x) \geq \alpha > 0$ and $b(x) \geq \beta > 0$.

Under these assumptions, the SPP (1) and (2) has a solution $u(x) \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^1(\Omega) \cap \mathcal{C}^2(\Omega^- \cup \Omega^+)$, when $\mu = 1$ the problem is the well known convection-diffusion problem [16] and when $\mu = 0$, we get the reaction-diffusion problem [14,17]. In the present study the following cases are dealt with

Case (i): $\sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}$ and

Case (ii): $\sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}$, where $\rho = \min\limits_{\overline{\Omega}} \left\{ \dfrac{b(x)}{a(x)} \right\}$.

Throughout this study C denotes a generic positive constant independent of nodal points, mesh size (N) and the perturbation parameters ε, μ. All functions in the continuous maximum norm are denoted by

$$\|w\|_{\overline{D}} = \max_{x \in \overline{D}} |w(x)|,$$

where $\overline{D}$ is a bounded closed interval $[r, s]$. The discrete maximum norm is defined as

$$\|W\|_{\overline{D}^N} = \max_{0 \leq i \leq N} |W(x_i)|,$$

where $\overline{D}^N$ is an arbitrary mesh on $\overline{D}$. If the domain is evident, it may be simply denoted as $\|.\|$ since the domain $\overline{D}^N$ is dropped.

The structure of the paper is as follows. In Section 2, we establish an existence theorem for (1) and (2), comparison principle, stability result and some *priori* estimates on the solution and its derivatives. Section 3 presents a hybrid finite difference scheme to solve the discrete problem, which generates robust numerical approximation to the solution. A decomposition of the discrete solution is introduced and truncation error analysis is estimated in Section 4. This analysis gears the main theoretical results presented. $\varepsilon - \mu$ uniform convergence in the maximum norm of the approximations are generated by the numerical method in Section 5. Numerical examples are provided in Section 6 to illustrate the applicability of the method with maximum pointwise errors and rate of convergence in the form of tables and graphs. The major finding of the paper is presented in conclusion.