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An entropy stable finite volume scheme for the two dimensional Navier–Stokes equations on triangular grids

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ABSTRACT

We construct a finite volume scheme for the compressible Navier–Stokes equations on triangular grids which are entropy stable at the semi-discrete level. This is achieved by using entropy stable inviscid fluxes constructed in the recently published work titled *Entropy Stable Scheme on Two-Dimensional Unstructured Grids for Euler Equations* by Ray, Chandrashekar, Fjordholm and Mishra, (2016), and computing viscous fluxes in terms of entropy variables. Wall boundary conditions are also constructed to be entropy stable and are imposed in a weak manner. The resulting scheme is applied to solve several standard viscous test cases, such as flow over a flat-plat, flow past a NACA-0012 airfoil and unsteady flow past a cylinder, to demonstrate its stability and accuracy.

1. Introduction

Finite volume methods are commonly used for solving convection-dominated problems, including compressible flows at high Reynolds numbers. These methods can be applied on unstructured grids, which makes them useful for problems involving complex domains. At a minimum, we require the numerical schemes to be stable and consistent. For the compressible Euler and Navier–Stokes equations, which are a system of non-linear hyperbolic-parabolic type of equations, the stability analysis can be based on the entropy condition and leads to non-linear stability [2].

A general approach to construct entropy stable schemes was introduced by Tadmor [3,4], which involved the construction of an entropy conservative scheme satisfying an entropy equality relation (valid for smooth solutions), followed by the addition of numerical dissipation to satisfy the requisite entropy stability condition at the discrete level. This idea has been used to construct entropy stable finite volume schemes for many important systems of conservation laws such as the shallow-water equations [5,6], the Euler equations [1,4,7–9] and the magnetohydrodynamics equations [10–12]. Entropy stable discontinuous Galerkin schemes for conservative systems have been constructed in [13–15].

The Navier–Stokes equations when written in terms of the *entropy variables* based on a specific choice of the entropy function, leads to the symmetrization of the inviscid and viscous flux Jacobians [2]. This property has been utilized to construct a finite-difference scheme for the Navier–Stokes equation in [16], and time-discontinuous Galerkin finite-element methods in [17]. An alternate approach of the Summation-by-Parts (SBP) framework has been used to derive provably stable, polynomial-based spectral collocation element methods of arbitrary order [18].

The prescription of well-posed boundary conditions for partial differential equations is of crucial importance. Most existing approaches are based on linearizing the Navier–Stokes equation near the boundary, followed by an energy method to

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derive suitable boundary conditions [19–21]. Nordström and Svärd [22] have used this idea to analyze the well-posedness of boundary conditions for the linearized Navier–Stokes system in three dimensions on a general domain. Svärd and Mishra [23] have constructed a conservative finite difference scheme using the SBP approach and simultaneous-approximation-term (SAT) penalty technique for the Euler equations on bounded domains, which have been shown to satisfy an appropriate *boundary entropy inequality* [24] numerically. Unfortunately, this methodology cannot be extended to the Navier–Stokes equation, as the specific form of entropy function used does not symmetrize the viscous fluxes. In [16], a normalized entropy function is used to derive a global energy estimate, with boundary conditions prescribed to bound/dissipate the total energy of the Navier–Stokes equations. However, it is not clear how one can consistently choose the various constants introduced to describe the inflow/outflow conditions. Recently, non-linear entropy-stable wall boundary conditions have been proposed in [25] and tested in the framework of discontinuous spectral collocation operators. The slip boundary condition for the Euler equations is imposed using a manufactured boundary state, the boundary viscous heat flux requires the construction of a suitable numerical boundary flux and the no-slip boundary condition is imposed using a standard SAT approach.

First-order entropy stable finite volume schemes for systems of hyperbolic conservation laws on unstructured meshes were analyzed in [26], and suitable error estimates were proved by assuming the mesh to be quasi-uniform. A first-order cell-centered entropy stable scheme for the two-dimensional Euler equations, was introduced and tested in [27]. A second-order vertex-centered entropy stable finite volume scheme for the Euler equations was proposed in [1], with the underlying unstructured primary grid consisting of triangular cells. In the present work, the scheme introduced in [1] is extended for the initial-boundary-value problem for the compressible Navier–Stokes equations in two dimensions. The inviscid flux is discretized at each control interface using an entropy stable flux [1,4,7,8], while the viscous fluxes are evaluated on triangles in terms of the entropy variables to preserve the symmetric structure of the continuous system. The boundary conditions are weakly imposed by constructing suitable inviscid boundary fluxes based on the numerical value at the boundary node and the given boundary data. Additionally, the gradient of entropy variables evaluated in boundary cells are corrected using the boundary data, which in turn ensures the proper evaluation of viscous fluxes. The above ingredients together lead to the derivation of discrete non-linear entropy estimates for the Navier–Stokes equations with homogeneous boundary conditions. The resulting semi-discrete scheme, which constitutes a set of ordinary differential equations (ODE), satisfies a discrete version of the entropy condition on any grid. In order to obtain a fully-discrete scheme, the time integration of the ODE system is performed with Runge–Kutta schemes. Although the analysis in this paper is performed in two-dimensions, it can easily be extended to the three-dimensional Navier–Stokes equations on tetrahedral grids.

The rest of the paper is structured as follows. In Section 2 we describe the Navier–Stokes equations. The entropy framework is discussed in Section 3, with global stability estimates for homogeneous boundary conditions. The discretization of the domain and the semi-discrete scheme is introduced in Section 4. Section 6 is the main section, where the discrete global entropy relation is obtained and the boundary fluxes are prescribed to obtain global stability results for homogeneous boundary conditions. Several standard numerical results are presented in Section 7, with concluding remarks in Section 8.

2. Navier–Stokes equations

Consider the following initial boundary value problem for the Navier–Stokes equations

$$\begin{aligned} \partial_t \mathbf{U} + \partial_x \mathbf{f}_1(\mathbf{U}) + \partial_y \mathbf{f}_2(\mathbf{U}) &= \partial_x \mathbf{g}_1(\mathbf{U}) + \partial_y \mathbf{g}_2(\mathbf{U}) \quad \forall\ \mathbf{x} = (x, y) \in \Omega \subset \mathbb{R}^2,\ t \in \mathbb{R}^+ \\ \mathbf{U}(\mathbf{x}, 0) &= \mathbf{U}_0(\mathbf{x}) \quad \forall\ \mathbf{x} \in \Omega + \textit{Boundary Conditions}. \end{aligned} \tag{1}$$

In the above equations, the conserved variables $\mathbf{U}$ and the inviscid fluxes $\mathbf{f}_1$, $\mathbf{f}_2$ are given by

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ E \end{pmatrix}, \quad \mathbf{f}_1(\mathbf{U}) = \begin{pmatrix} \rho u_1 \\ \rho u_1^2 + p \\ \rho u_1 u_2 \\ (E+p) u_1 \end{pmatrix}, \quad \mathbf{f}_2(\mathbf{U}) = \begin{pmatrix} \rho u_2 \\ \rho u_2 u_1 \\ \rho u_2^2 + p \\ (E+p) u_2 \end{pmatrix},$$

while the viscous fluxes $\mathbf{g}_1$, $\mathbf{g}_2$ are given by

$$\mathbf{g}_1(\mathbf{U}) = \begin{pmatrix} 0 \\ \tau_{11} \\ \tau_{12} \\ u_1 \tau_{11} + u_2 \tau_{12} - Q_1 \end{pmatrix}, \quad \mathbf{g}_2(\mathbf{U}) = \begin{pmatrix} 0 \\ \tau_{21} \\ \tau_{22} \\ u_1 \tau_{21} + u_2 \tau_{22} - Q_2 \end{pmatrix}.$$

Here ρ, $\mathbf{u} = (u_1, u_2)^\top$ and p denote the fluid density, velocity and pressure respectively. The quantity E is the total energy per unit volume given by $E = \rho(e + |\mathbf{u}|^2/2)$, where e is the specific internal energy given by a caloric equation of state, $e = e(\rho, p)$. For an ideal gas, $e = p/(\gamma - 1)\rho$ with $\gamma = c_p/c_v$ denoting the ratio of specific heats. In the viscous fluxes, the shear stress tensor $\boldsymbol{\tau}$ and the heat flux $\mathbf{Q}$ are given by Newtonian and Fourier constitutive relations, respectively,

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} = \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top) - \frac{2}{3}\mu(\nabla \cdot \mathbf{u})\mathcal{I}, \quad \mathbf{Q} = (Q_1, Q_2) = -\kappa \nabla \theta,$$

where $\mathcal{I}$ is the unit tensor, μ is the coefficient of dynamic viscosity, κ is the coefficient of heat conductance and θ is the absolute fluid temperature. Furthermore, θ is obtained using the *ideal gas law* given by $p = \rho R \theta$, where R is the gas constant,