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Topological aspects of weighted graphs with application to fixed point theory

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ABSTRACT

In this work, we introduce new concepts of G-monotone sequences, G-bounded and G_τ-compact nonempty subsets of the set of vertices of a weighted digraph G, where τ is a sequential convergence. We also provide an application to metric fixed point theory.

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1. Introduction

Following the publication of Ran and Reurings [13] of their extension of the Banach Contraction Principle to partially ordered metric spaces, many mathematicians were interested into this new area of research within the metric fixed point theory. It was Jachymski [8] who gave a more general formulation of these results by considering digraphs instead of a partial order. Since then, many publications appeared in this new direction which bridges the graph theory and the metric fixed point theory, see for instance [1,2]. Their approach is to define a digraph on a metric space. Then based on the properties of the metric space, they were able to prove some fixed point results. In this work, we take a different approach. Indeed, we consider a weighted digraph and introduce some needed topological structures on the set of vertices. These properties are analogues to the ones used in metric spaces. We also show that in fact they are not equivalent. Finally we are able to prove some fixed point results which are more general than the ones found in the literature.

In this paper, we study weighted digraphs. A weighted digraph is a digraph that has a numeric label to each edge. Such labels can be integers, rational numbers, or real numbers, which represent a concept such as distance, connection costs, or affinity. For example, if we use a graph to represent the roads between cities, and if we are interested to find the fastest way to travel cross-country, then it is not appropriate for all edges to be equal since some intercity distances will likely be much larger than others. Thus it is natural to consider graphs whose edges are not weighted equally.

The bridge between the graph theory and the fixed point theory is motivated by the fact that they often arise in industrial fields such as image processing engineering, physics, computer science, economics, ladder networks, dynamic programming, control theory, stochastic filtering, statistics, telecommunication and many other applications. To the best of our knowledge, the approach presented here is new and has never been carried out.

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2. Topological aspects of weighted graphs

A graph G consists of a nonempty set $V(G)$ of objects called vertices together with a possibly empty set $E(G)$ of unordered pairs of vertices; the elements of $E(G)$ are called edges. If a direction is imposed on each edge, we call the graph a directed graph or digraph. Digraphs can have two edges with the same endpoints, provided they have opposite directions. The underlying graph $\widetilde{G}$ of a digraph is constructed by ignoring all directions and replacing any resulting multiple edges by single edges. We assume that our concerned digraphs are reflexive i.e., each vertex has a loop. Moreover, we will say that a digraph G is transitive if

$$(f, g) \in E(G) \quad \text{and} \quad (g, h) \in E(G) \Rightarrow (f, h) \in E(G) \quad \text{for all} \quad f, g, h \in V(G).$$

Throughout this work, we consider weighted digraphs where the weight of each edge is given by a distance function between vertices. Moreover, we will use the concept of increasing or decreasing sequences in the sense of a digraph. Therefore, the following definition is needed.

Definition 2.1. Let G be a digraph. A sequence $(x_n) \in V(G)$ is said to be

(a) G-increasing if $(x_n, x_{n+1}) \in E(G)$, for all $n \in \mathbb{N}$;
(b) G-decreasing if $(x_{n+1}, x_n) \in E(G)$, for all $n \in \mathbb{N}$;
(c) G-monotone if (x_n) is either G-increasing or G-decreasing.

Let G be a digraph. Next we define the concept of G-compactness. In order to do this, we will need some kind of sequential convergence in $V(G)$. For example, if τ is a topology on $V(G)$, then we may consider the τ-convergence of a sequence. But there are sequential convergence that are not necessarily generated by a topology. We will still use the notation τ-convergence without assuming that τ is a topology.

Definition 2.2. Let G be a digraph and τ be as described above. A nonempty subset C of $V(G)$ is said to be G_τ-compact if any G-increasing (resp. G-decreasing) sequence $(x_n) \in C$ has a subsequence $(x_{\phi(n)})$ which is τ-convergent to an element x in C and $(x_{\phi(n)}, x) \in E(G)$ (resp. $(x, x_{\phi(n)}) \in E(G)$), for every $n \in \mathbb{N}$. In particular, if G is transitive, then we will have $(x_n, x) \in E(G)$, (resp. $(x, x_n) \in E(G)$) for every $n \in \mathbb{N}$.

Using the standard definition of τ-compactness in a metric space, we will say that a nonempty subset C of $V(G)$ is τ-compact if and only if any sequence in C has a subsequence which τ-converges to a point in C. Note that if G is transitive and the G-intervals are τ-closed, then any τ-compact subset C of $V(G)$ is G_τ-compact.

Example 2.1. Consider the family of intervals $(I_s)_{s\in[0,+\infty)}$, in $\mathbb{R}^2$ defined by

$$I_s = \{(x, y)\ ;\ x = s \text{ and } 0 \le y \le \lceil s \rceil + 1\}.$$

On $\mathbb{R}^2$ define the digraph G by $((x, y), (a, b)) \in E(G)$ if and only if (x, y) and (a, b) belong to some I_s, for $s \in [0, +\infty)$, and $y \le b$. It is clear that if $((x_n, y_n))$ is a G-monotone sequence, then there exists $s_0 \in [0, +\infty)$ such that $(x_n, y_n) \in I_{s_0}$, for all $n \in \mathbb{N}$. If τ is the Euclidean topology, then $\mathbb{R}^2$ is G_τ-compact and any G-monotone sequence is bounded.

This example suggests the following definition.

Definition 2.3. Let G be a weighted digraph. Let d be a metric distance on $V(G)$. A nonempty $C \subseteq V(G)$ is said to be weakly G-bounded if any G-monotone sequence $(x_n)_{n\in\mathbb{N}}$ in C is bounded, i.e. $\delta((x_n)) = \sup\limits_{n,m\in\mathbb{N}} d(x_n, x_m) < \infty$.

In fact it is clear that the motivation behind introducing G-compactness is due to first the sequential characterization of metric compact sets and the use of monotone sequences in the study of fixed points of monotone mappings. Therefore, whenever a topological concept is characterized by sequences, it will have a similar extension into weighted graphs. For example, we have the following definition.

Definition 2.4. Let G be a weighted digraph. Let d be a metric distance on $V(G)$. A nonempty $C \subseteq V(G)$ is said to be G-complete (or a G-Cauchy space) if every G-monotone Cauchy sequence of vertices in C has a limit that is also in C.

Remark 2.1. It is amazing that in the Ran and Reurings extension [13] of the Banach Contraction Principle to partially metric spaces, one will only need to assume Order-completeness in the sense that monotone Cauchy sequences are convergent. Now one may ask whether such completeness is different from the metric completeness. A small modification of Example 2.1 will settle this question.

Example 2.2. Consider the set $C = \{(x, y) \in \mathbb{R}^2;\ 0 \le x < 1 \text{ and } 0 \le y \le 1\}$. Consider the family of intervals $(I_s)_{s\,\in\,[0,\,1)}$, in C defined by

$$I_s = \{(x, y)\ ;\ x = s \text{ and } 0 \le y \le 1\}.$$

On C define the digraph G by $((x, y), (a, b)) \in E(G)$ if and only if (x, y) and (a, b) belong to some I_s, for $s \in [0, 1)$, and $y \le b$. It is clear that if $((x_n, y_n))$ is a G-monotone sequence, then there exists $s_0 \in [0, 1)$ such that $(x_n, y_n) \in I_{s_0}$, for all $n \in \mathbb{N}$. If τ is the Euclidean topology, then C is G-complete but not complete.

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