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Nonlocal symmetries and explicit solutions for the Gardner equation

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ABSTRACT

From the known Lax pair of the Gardner equation, the Lie symmetry group method is successfully applied to find exact invariant solutions for the Gardner equation with nonlocal symmetries by introducing suitable auxiliary dependent variables. Based on the prolonged system, the explicit analytic solutions related to the Jacobi elliptic functions are derived. Figures illustrate the interaction between the cnoidal waves and a kink wave.

1. Introducton

The Gardner equation (GE), or the combined KdV and modified-KdV equation for the description of long nonlinear internal waves is [1–7]

$$u_t + \alpha u u_x + \beta u^2 u_x + \gamma u_{xxx} = 0, \tag{1}$$

which is written in dimensionless form. It exhibits basically the same properties as the classical KdV, but extends its range of validity to a wider interval of parameters of internal wave motion for a given environment [8–15]. In Eq. (1), $u(x, t)$ is a wave function, x and t are the space and time coordinates, respectively. In the most applications, $u = u(x, t)$ represents the perturbation of an isopycnal surface of the relevant wave mode, the terms uu_x and u^2u_x represent the nonlinear wave steepening and the third-order derivative term u_{xxx} represents the dispersive wave effects. The coefficients of the nonlinear terms α and β as well as the dispersive term γ are determined by the steady oceanic background density and flow stratification through the linear eigenmode of the internal waves [16,17].

The Gardner equation (1), like the Korteweg–de Vries equation, is also an integrable model. In addition, the GE appears in various fields such as hydrodynamics [18], theoretical physics [19–21] and plasma physics [22], its numerical studies could also be found [23–25]. Exact soliton solutions of the GE were given by several authors [26–28]. Then in 2006, Lou and Ma [29] put forward the direct method of symmetric transformation group for Lax integrable system instead the traditional method of solving symmetric transformation group. As we all know, the nonlocal symmetry has closed relation with the integrable model and is beneficial to the enlarge of the class of the symmetry which provides the chance of obtaining the exact solution. However, the nonlocal symmetry can not construct solution directly. In other words, only nonlocal symmetry is not enough unless we localized nonlocal symmetry into the local one with the closed prolonged system.

Recently, in view of the nonlocal symmetry of the nonlinear partial equations (NPDES), Lou and co-workers [30] obtained the explicit analytic interaction solutions between cnoidal waves and solitary waves for the well-known KdV equation which

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are related to Darboux transformation. Later, by studying the new exact solutions of the equations of the mKdV, ANKS and Boussinesq system, Xin and co-workers [31–33] further proved the effectiveness of the proposed method.

In this article, the structure of the layout is as follows: in Section 2, under the infinitesimal transformation of variable u, a vector symmetry which contains the classical Lie point symmetry and the nonlocal symmetry is derived directly. In Section 3, with the aid of the Lax pair of the GE, the nonlocal symmetry $\psi_1 = \psi_x$ is localized in the properly prolonged system by introduction two suitable auxiliary variables. As a result, the finite transformation and the general Lie point symmetry of the prolonged system can be obtained. In Section 4, on the basis of the prolonged system, we get an optimal system of the GE and some reductions, thus some new explicit solutions are constructed. The different waves structure pictures give a visual representation of these solutions. Section 5 is a simple conclusion of this paper.

2. Nonlocal symmetry of the GE

The Lax pair of GE (1) is

$$\psi_{xx} + \frac{1}{A\gamma}(2\beta u + \alpha)\psi_x = 0, \tag{2}$$

$$\psi_t - \frac{\alpha}{A}\psi_{xx} + \frac{1}{3}(\beta u^2 + 2\alpha u + A\gamma u_x)\psi_x = 0, \tag{3}$$

under the compatibility condition $\psi_{xx,t} = \psi_{t,xx}$, where $A = \pm\sqrt{-6\beta/\gamma}$. To seek the nonlocal symmetries, we use a method that can obtain the nonlocal symmetries directly. The fact shows that this method not only can obtain the nonlocal symmetries but also the general Lie point symmetries of the given equations.

First, the symmetry of the GE is defined as a solution of its linearized equation

$$\sigma_{1,t} + \gamma\sigma_{1,xxx} + \alpha u\sigma_{1,x} + \alpha\sigma_1 u_x + \beta u^2\sigma_{1,x} + 2\beta\sigma_1 uu_x = 0, \tag{4}$$

which means that Eq. (5) is form invariant under the transformation

$$u \to u + \varepsilon\sigma_1, \tag{5}$$

with the infinitesimal parameter ε.

The symmetry can be written as

$$\sigma_1 = \xi(x, t, u, \psi, \psi_x)u_x + \tau(x, t, u, \psi, \psi_x)u_t - U(x, t, u, \psi, \psi_x). \tag{6}$$

This assumption shows that this kind of symmetry is neither a classical Lie point symmetry nor a Lie–Bäcklund symmetry because it depends on the auxiliary variables and their high order partial derivatives. As a result by substituting Eq. (6) into Eq. (4) and solving the determining equations, the vector symmetry can be derived

$$\sigma_1 = \left(\frac{c_1 x}{3} - \frac{c_1\alpha^2 t}{6\beta} + c_3\right)u_x + (c_1 t + c_2)u_t + \frac{c_1}{3}u + \frac{c_1\alpha}{6\beta} - c_4\psi_x, \tag{7}$$

where $c_i (i = 1, \dots, 4)$ are four arbitrary constants. Eq. (7) contains the classical Lie point symmetry $\sigma_{11} = (\frac{c_1 x}{3} - \frac{c_1\alpha^2 t}{6\beta} + c_3)u_x + (c_1 t + c_2)u_t + \frac{c_1}{3}u + \frac{c_1\alpha}{6\beta}$ and the nonlocal symmetry $\sigma_{12} = -c_4\psi_x$.

3. Localization of the nonlocal symmetry

For the vector symmetry (7), letting $c_1 = c_2 = c_3 = 0$ and $c_4 = -1$, we have the nonlocal organization

$$\sigma_1 = \psi_x. \tag{8}$$

We know that nonlocal symmetries cannot be directly employed to construct explicit solutions. Hence, nonlocal symmetries need to be transformed into local ones. One may extend the original system to a closed prolonged system by introducing some additional dependent variables. At the same time, the linearized equations

$$A\gamma\sigma_{2,xx} + 2\beta u\sigma_{2,x} + 2\beta\sigma_1\psi_x + \alpha\sigma_{2,x} = 0, \tag{9a}$$

$$A\left[2\sigma_1(\alpha + \beta u)\psi_x + 3\sigma_{2,t} + (2\alpha u + \beta u^2)\sigma_{2,x}\right] - 3\alpha\sigma_{2,xx} - 6\beta u_x\sigma_{2,x} - 6\beta\sigma_{1,x}\psi_x = 0, \tag{9b}$$

are the direct results of Eqs. (2) and (3) respectively, under the symmetry transformation $\psi \to \psi + \varepsilon\sigma_2$.

In order to localization of the nonlocal symmetry (7), we introduce the following variable:

$$\psi_1 = \psi_x, \tag{10}$$

the corresponding linearized equations are

$$\sigma_{2,x} - \sigma_3 = 0, \tag{11}$$