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Least square ellipsoid fitting using iterative orthogonal transformations

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ABSTRACT

We describe a generalised method for ellipsoid fitting against a minimum set of data points. The proposed method is numerically stable and applies to a wide range of ellipsoidal shapes, including highly elongated and arbitrarily oriented ellipsoids. This new method also provides for the retrieval of rotational angle and length of semi-axes of the fitted ellipsoids accurately. We demonstrate the efficacy of this algorithm on simulated data sets and also indicate its potential use in gravitational wave data analysis.

1. Introduction

Reconstructing 3-dimensional (3D) ellipsoidal surfaces from discrete data points is a well-studied problem in the field of computer vision, pattern recognition, astronomy and medical image processing. Such 3D ellipsoid models find use in a broad class of applications: (a) One of the primary goals of computer vision is to find a suitable 3D shape descriptor for arbitrary shapes while preserving geometrical information as much as possible. Existing shape representations based on spherical descriptors are limited in scope as they are appropriate only for nearly spherical bodies. On the other hand, the ellipsoidal descriptors provide a closer approximation to irregular 3D shapes. Khatun et al. [1] have proposed a method where ellipsoidal 3D shape representations have been used to retrieve arbitrary 3D shapes. (b) Gait analysis is utilised for a systematic study of animal locomotion by measuring and tracking their body mechanics. Gathering gait features and interpretation of the gait dynamics is a challenging task as current techniques are computationally expensive. 3D ellipsoid fitting methods have been found useful in this line of research. Sivapalan et al. [2] have proposed a fast 3D ellipsoid based gait recognition algorithm using a 3D voxel model, (c) galaxies are often modelled as 3-dimensional ellipsoids whose parameters are determined from images recorded by telescopes. Compère et al. [3] have recently proposed a three-dimensional galaxy fitting algorithm to extract parameters of the bulge, long bar, disc and a central point source from broadband images of galaxies.

Several techniques exist for fitting ellipsoids to a set of data points and can be broadly classified into projection based algorithms [4,5] and nonlinear optimisation based surface fitting algorithms [6,7]. Projection based fitting algorithms are also further organised into two categories: namely, orthographic and line integral based.

The basic idea in orthographic projection is to use matrix operators to project a 3D shape onto planes. For the case of a 3D ellipsoid, three different orthographic transformations are possible along three orthogonal planes, and projected shapes are 2D ellipses. If the parameters of the projected ellipses are deciphered, then 3D rotation between two successive projections can be detected after characterising the variations of the semi-axes length and the orientation of the projected ellipses. On the other hand, the line integral projection based methods commonly use projection contours to reconstruct

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ellipsoids. The general second-degree equation of an ellipsoid is used to construct the line integral projection model. One of the shortcomings of such methods is that one needs prior information about the projected ellipses to determine the angle and axis of rotation accurately.

In nonlinear optimisation techniques for fitting ellipsoidal surfaces, the latter is modelled as a bounded surface through a family of polynomials which are then fitted using standard nonlinear optimisation methods. The problem with such techniques is that due to high non-linearity of the model, the optimal solution may get stuck in local solutions leaving the resulting surface unbounded. Therefore the proper solution cannot guarantee closed bounded solution of the desired surface. To overcome this problem Li and Griffiths [8] have prescribed an algorithm to obtain closed form of the resulting ellipsoidal surface by providing an additional constraint. The fitting algorithm works robustly for ellipsoids whose short radius is at least half of their major radius and for which the semi-axes of the model ellipsoid are aligned along the co-ordinates. More recently, Ying et al. [9] have proposed a least-square ellipsoid fitting algorithm by extending the 2-dimensional ellipse fitting algorithm given by Fitzgibbon et al. [10].

In this work, we propose a stable algorithm that can fit an ellipsoidal surface to a given set of data points and can detect the rotational angle as well as semi-axes length with significant improvement over Li and Griffiths [8]. The new method is applicable even to extreme cases where the ellipsoid is highly elongated and arbitrarily oriented to a rigid frame of reference. Our primary motivation is to extend their idea in such an algorithmic form, which can produce the best fit for *any kind* of the ellipsoidal surface. Also, we describe a method for the retrieval of the orientation of such ellipsoids without assuming any prior information.

This paper is organised as follows: In Section 2, we present a concise description of Li and Griffiths [8], establishing the notation used in this article and highlighting the salient features of their algorithm. Section 3 describes our proposed method based on the general equation of an ellipsoid. In Section 4, we describe the algorithm for retrieval of the orientation of the reconstructed ellipsoid, followed by a demonstration of the efficacy of our method using synthetic data. We then present a case study where this approach is applied to the field of gravitational wave data analysis in Section 5.

Finally, we make some general comments on the results obtained in this paper.

2. Previous work

The general equation of the second degree in three variables (x, y, z) representing a conic is given by:

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy + 2px + 2qy + 2rz + d = 0. \tag{1}$$

As shown in [8], Eq. (1) represents an ellipsoid under the constraint

$$kJ - I^2 = 1. \tag{2}$$

where

$$I \equiv a + b + c, \tag{3}$$

$$J \equiv ab + bc + ac - f^2 - g^2 - h^2, \tag{4}$$

and k is a positive number. For ellipsoids with comparable semi-axes lengths, $k \sim 4$.

Let P = $\mathbf{p_i}(\mathbf{x_i}, \mathbf{y_i}, \mathbf{z_i})$, $\{i = 1,2,..,N\}$ be the coordinates of N points with respect to a fixed frame of reference XYZ (refer Fig. 1b) to which an ellipsoid is to be fitted. Further, let the ellipsoid be arbitrarily oriented in this frame. For every point $\mathbf{p}_i$, one defines a column $\mathbf{X_i}$ of the *design matrix* $\mathbf{D}$ as:

$$\mathbf{X_i} = (x_i^2, y_i^2, z_i^2, 2y_iz_i, 2x_iz_i, 2x_iy_i, 2x_i, 2y_i, 2z_i, 1)^T. \tag{5}$$

To fit an ellipsoidal surface, each data point must satisfy the quadratic equation (1) with the constraint defined in Eq. (2). The algebraic distance Ω between the model and the set of data points defined as,

$$\Omega = \sum_{i=1}^{N} (\mathbf{v}^T\mathbf{X_i})^2 \tag{6}$$

must be minimized with respect to $\mathbf{v}$ in order to find the best fit, where

$$\mathbf{v} \equiv (a, b, c, f, g, h, p, q, r, d)^T \tag{7}$$

is the set of unknown parameters. Therefore the ellipsoid fitting problem can be mapped to an optimisation problem that can be solved using standard least square methods.

It is obvious that Eq. (1) can be written in *matrix form* as the following system of linear equations:

$$\mathbf{D}^T\mathbf{v} = 0, \tag{8}$$

in terms of the design matrix $\mathbf{D} = (X_1, X_2, ..., X_i)$ of order $10 \times N$, where $N \geq 10$. The geometric distance above can also be written in matrix form as $\Omega = ||\mathbf{Dv}||^2 = \mathbf{v}^T\mathbf{D}^T\mathbf{vD}$, which is to be minimised subject to the constraint given in Eq. (2). The