Applied Mathematics and Computation 314 (2017) 429–438

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Mixed metric dimension of graphs

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ARTICLE INFO

MSC:
05C12
05C85
05C90

Keywords:
Mixed metric dimension
Edge metric dimension
Metric dimension

ABSTRACT

Let $G=(V,E)$ be a connected graph. A vertex $w\in V$ distinguishes two elements (vertices or edges) $x, y\in E\cup V$ if $d_G(w,x)\neq d_G(w,y)$. A set S of vertices in a connected graph G is a mixed metric generator for G if every two distinct elements (vertices or edges) of G are distinguished by some vertex of S. The smallest cardinality of a mixed metric generator for G is called the mixed metric dimension and is denoted by $\dim_{\mathrm{m}}(G)$. In this paper we consider the structure of mixed metric generators and characterize graphs for which the mixed metric dimension equals the trivial lower and upper bounds. We also give results about the mixed metric dimension of some families of graphs and present an upper bound with respect to the girth of a graph. Finally, we prove that the problem of determining the mixed metric dimension of a graph is NP-hard in the general case.

1. Introduction

Given a simple and connected graph $G=(V,E)$ and two vertices $x, y\in V$, the distance $d_G(x, y)$ (or $d(x, y)$ for short) between x and y is the length of a shortest $x-y$ path. A vertex $v\in V$ is said to *distinguish* (we also use the terms "recognize" or "determine" instead of "distinguish") two vertices x and y, if $d_G(v,x)\neq d_G(v,y)$. A set $S\subset V$ is called a *metric generator* for G if any pair of distinct vertices of G is distinguished by some element of S. A metric generator of minimum cardinality is a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$.

The concept of metric dimension was introduced by Slater in [24], where the metric generators were called *locating sets*, according to some connection with the problem of uniquely recognizing the position of intruders in networks. On the other hand, the concept of metric dimension of a graph was independently introduced by Harary and Melter in [14], where metric generators were named *resolving sets*. After these two seminal papers, several works concerning applications, as well as some theoretical properties, of this invariant were published. For instance, applications to the navigation of robots in networks are discussed in [17] and applications to chemistry in [5,6,15]. Furthermore, this topic has found some applications to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [19].

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1 This author is financed in part by ARRS Slovenia under the grant P1-0297.

http://dx.doi.org/10.1016/j.amc.2017.07.027
0096-3003/

Some interesting connections between metric generators in graphs and the Mastermind game or coin weighing have been presented in [4].

On the other hand, with respect to the theoretical studies on this topic, different points of view of metric generators have been described in the literature, which have highly contributed to gain more insight into the mathematical properties of this parameter related with distances in graphs. Several authors have introduced other variations of metric generators like for instance, resolving dominating sets [2], independent resolving sets [7], local metric sets [21], strong resolving sets [20], simultaneous metric generators [23], k-metric generators [10,26], resolving partitions [8], strong resolving partitions [13], k-antiresolving sets [25], etc. have been presented and studied.

Moreover, several interesting articles concerning metric dimension of graphs can be found in the literature. However, according to the amount of results on this topic, we prefer to cite only those papers which are important from our point of view. In concordance with it, we refer the reader to the work [1], where it can be found some historical evolution, nonstandard terminologies and more references on this topic, and the recent work [11], where a general approach on metric generators is described. Some other interesting results and a high number of references can be found in the theses [9,18,22].

In connection with describing other new variants of metric generators in graph, very recently a parameter used to uniquely recognize the edges of the graph has been introduced in [16]. Roughly speaking, there was used a graph metric to identify each pair of edges by mean of distances to a fixed set of vertices. This was based on the fact that a metric basis S of a connected graph G uniquely identifies all the vertices of G by mean of distance vectors, but not necessarily such metric basis uniquely recognizes all the edges of the graph. In this sense, the following concepts deserved to be considered.

Given a connected graph $G = (V, E)$, a vertex $v \in V$ and an edge $e = uw \in E$, the distance between the vertex v and the edge e is defined as $d_G(e, v) = \min\{d_G(u, v), d_G(w, v)\}$. A vertex $x \in V$ *distinguishes* (*recognizes* or *determines*) two edges e_1, $e_2 \in E$ if $d_G(x, e_1) \neq d_G(x, e_2)$. A set S of vertices in a connected graph G is an *edge metric generator* for G if every two distinct edges of G are distinguished by some vertex of S. The smallest cardinality of an edge metric generator for G is called the *edge metric dimension* and is denoted by $\dim_e(G)$. An *edge metric basis* for G is an edge metric generator for G of cardinality $\dim_e(G)$.

Having defined the concept of edge metric generator, which uniquely determines every edge of the graph, one could think that probably any edge metric generator S is also a standard metric generator, *i.e.* every vertex of the graph is identified by S or vice versa. However, as it was proved in [16], this is further away from the reality, although there are several graph families in which such facts occur. In [16], among other results, some comparison between these two parameters above were discussed. As a consequence of the study, families of graphs G, for which $\dim_e(G) < \dim(G)$ or $\dim_e(G) = \dim(G)$ or $\dim(G) < \dim_e(G)$ hold were described.

In the present work we focus in a kind of mixed version of these two parameters described above. That is, given a connected graph G, we wish to uniquely identify the elements (edges and vertices) of G by means of vector distances to a fixed set of vertices of G.

Since the (edge or mixed) metric dimension is defined only over connected graphs, in order to avoid repetitions, from now on in this article, all the graphs which will be considered are connected, even so we do not explicitly mention it. Moreover, we do not consider here any graph with only one vertex (a singleton). That is, from now on, all the studied graphs contain at least two vertices.

In the next section, we formally define mixed metric dimension of a graph and present an equivalent definition of the problem in the form of a linear program. Further, we study the structure of mixed metric generators. We present necessary conditions for a vertex to be included in a mixed metric generator. Moreover, we characterize graphs with extreme mixed metric dimensions (2 or number of vertices). In Section 4, we present results about the mixed metric dimension of several families of graphs. Section 5 is used to give an upper bound for the mixed metric dimension of a graph with respect to the girth of the graph. Finally, in Section 6 we study the complexity of the problem of determining the mixed metric dimension of a graph and show that it is NP-hard in general. We conclude the paper with six open problems.

2. Definition of the problem

We say that a vertex v of a connected graph G *distinguishes* two elements (vertices or edges) x, y of G if $d_G(x, v) \neq d_G(y, v)$. A set S of vertices of G is a *mixed metric generator* if any two distinct elements (vertices or edges) of G are distinguished by some vertex of S. The smallest cardinality of a mixed metric generator for G is called the *mixed metric dimension* and is denoted by $\dim_m(G)$. A *mixed metric basis* for G is a mixed metric generator for G of cardinality $\dim_m(G)$.

The problem of determining the mixed metric dimension of a given graph can also be restated as the following optimization problem. Let us now present this mathematical programming model which can be used to solve the problem of computing the mixed metric dimension or finding a mixed metric basis for a graph G. A similar model for the case of the standard metric dimension was described in [5].

Let G be a graph of order n and size m with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{e_1, e_2, \dots, e_m\}$. We consider the $n \times (n+m)$ dimensional matrix $D = [d_{ij}]$ such that $d_{ij} = d_G(x_i, x_j)$ and $x_i \in V$ and $x_j \in V \cup E$. Now, given the variables $y_i \in \{0, 1\}$ with $i \in \{1, 2, \dots, n\}$ we define the following function:

$$\mathcal{F}(y_1, y_2, \dots, y_n) = y_1 + y_2 + \cdots + y_n.$$