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Neighbor sum distinguishing total chromatic number of planar graphs with maximum degree 10

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ABSTRACT

Given a simple graph G, a proper total-k-coloring $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ is called *neighbor sum distinguishing* if $S_\phi(u) \neq S_\phi(v)$ for any two adjacent vertices $u, v \in V(G)$, where $S_\phi(u)$ is the sum of the color of u and the colors of the edges incident with u. It has been conjectured by Pilśniak and Woźniak that $\Delta(G) + 3$ colors enable the existence of a neighbor sum distinguishing total coloring. The conjecture is confirmed for any graph with maximum degree at most 3 and for planar graph with maximum degree at least 11. We prove that the conjecture holds for any planar graph G with $\Delta(G) = 10$. Moreover, for any planar graph G with $\Delta(G) \geq 11$, $\Delta(G) + 2$ colors guarantee such a total coloring, and the upper bound $\Delta(G) + 2$ is tight.

1. Introduction

A multigraph is *irregular* if no degrees of two vertices are equal. In an arbitrary simple graph, there exist two vertices with the same degree. The commonly known basic fact has brought forth a number of implicative questions. In particular an issue of a possible definition of an irregular graph was raised by Chartrand [3]. A multigraph can be viewed as a weighted simple graph with nonnegative-integer weights on the edges. That is, the weighting $c_1 : E(G) \to \{1, 2, \ldots, k\}$, assigning every edge an integer corresponding to its multiplicity in a desired multigraph, where by $d_{c_1}(v) = \sum_{uv \in E(G)} c_1(uv)$ we denote the *degree* of $v \in V(G)$. The least k such that there do not exist two different vertices with the same degree in such a coloring of the simple graph G is called *irregularity strength* of G. The irregular strength was studied in numerous papers, e.g., [13,15]. Moreover, it is the cornerstone of many graph invariants and a new general direction in research on vertex distinguishing graph colorings.

In this paper, we consider the following interesting problem. Given a simple graph G and a proper total-k-coloring $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$. For every $v \in V$ denote by $S_\phi(v)$ the sum of the colors of edges incident with v and the color of v. If $S_\phi(u) \neq S_\phi(v)$ holds for every edge $uv \in E(G)$, then ϕ is *total neighbor sum distinguishing*, or *tnsd-k-coloring* for simplicity. If $uv \in E(G)$ and $S_\phi(u) = S_\phi(v)$, then we say u *conflicts* with v with respect to ϕ. The *neighbor sum distinguishing total chromatic number* $\chi''_\Sigma(G)$ of G is the least k which guarantees the existence of a tnsd-k-coloring for G. Pilśniak and Woźniak put forward the following conjecture.

Conjecture 1.1 [14]**.** *For any graph G, $\chi''_\Sigma(G) \leq \Delta(G) + 3$.*

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It is worth mentioning that the concept investigated in this article was also inspired by another famous problem in the field of vertex distinguishing colorings of graphs. Given a total-k-coloring ϕ of G, $C_\phi(u)$ is the set of the color of vertex u and the colors of the edges incident with u. We say ϕ is *total adjacent vertex distinguishing* or *total-avd-k-coloring* for G if $C_\phi(u) \neq C_\phi(v)$ for any two adjacent vertices $u, v \in V(G)$, and we write $\chi''_a(G)$ for the least k that guarantees the existence of such a total-avd-k-coloring. In 2005, Zhang [21] proposed the following conjecture.

Conjecture 1.2 [21]**.** *For any graph G with at least two vertices,* $\chi''_a(G) \leq \Delta(G) + 3$.

Coker and Johannson [5] used a probabilistic method to establish an upper bound $\Delta + C$ for $\chi''_a(G)$, where C is a positive constant. Conjecture 1.2 was confirmed for planar graphs with large maximum degree [20].

The Conjecture 1.1 might be viewed as a significant generalization of Conjecture 1.2. Pilśniak and Woźniak [14] proved that Conjecture 1.1 holds for complete graphs, cycles, bipartite graphs and subcubic graphs. Applying the Combinatorial Nullstellensatz, Ding et al. [6] proved that $\chi''_\Sigma(G) \leq 2\Delta(G)+\text{col}(G) - 1$, where col($G$) is the color number of G. Later Ding et al. [7] improved it to $\chi''_\Sigma(G) \leq \Delta(G) + 2\text{col}(G) - 2$. Recently, Przybyło proved that $\chi''_\Sigma(G) \leq \Delta(G) + \lceil \frac{5}{3}\text{col}(G)\rceil$ in [16]. Loeb and Tang [12] proved this bound to be asymptotically correct by showing that $\chi''_\Sigma(G) \leq (1 + o(1))\Delta(G)$. Meanwhile, Li et al. [10] proved that Conjecture 1.1 holds for planar graphs with $\Delta(G) \geq 13$. Qu et al. extended it to the list version in [17]. For more results, the reader is referred to [8,9,11]. Recently, Qu et al. proved that Conjecture 1.1 holds for planar graphs with $\Delta(G) = 11$ and $\Delta(G) = 12$ [18]. Cheng et al. proved a stronger conclusion in the following theorem.

Theorem 1.1 [4]**.** *For any planar graph G with maximum degree* $\Delta(G) \geq 14$, $\chi''_\Sigma(G) \leq \Delta(G) + 2$.

Meanwhile, Song et al. considered the case when $\Delta(G) \geq 12$, and obtained the following result.

Theorem 1.2 [19]**.** *For any planar graph G with maximum degree* $\Delta(G) \geq 12$, $\chi''_\Sigma(G) \leq \Delta(G) + 2$.

It is straightforward to see that the upper bound $\Delta(G) + 2$ in Theorems 1.1 and 1.2 is tight if there exist two adjacent vertices with maximum degree in G.

In this paper, we provide a completely new argument proving a new result simultaneously improving all known bounds.

Theorem 1.3. *For any planar graph G with maximum degree* $\Delta(G)$, $\chi''_\Sigma(G) \leq \max\{\Delta(G) + 2, 13\}$.

Theorem 1.3 implies Theorem 1.1 and the following corollary.

Corollary 1.1. *For any planar graph G with maximum degree 10,* $\chi''_\Sigma(G) \leq 13$.

2. Preliminaries

For all terminologies and notations used but undefined in this paper, we follow [2]. Given a plane graph G on the Euclidean plane, we write $F(G)$ for the face set of G. For simplicity, we shall refer to a vertex of degree k (at least k, at most k) as a k-vertex (k^+-vertex, k^--vertex) and a face of degree k (at least k, at most k) as a k-face (k^+-face, k^--face), respectively. We write $n^G_k(u)$ for the number of k-vertices adjacent to u, and analogically we define $n^G_{k+}(u), n^G_{k-}(u)$. A k-vertex v is a *bad* k-neighbor of u if the edge uv is incident with two 3-faces, and v is a *special* k-neighbor of u if the edge uv is incident with exactly one 3-face. Analogically, we write $n^G_{kb}(u)$ and $n^G_{ks}(u)$ for the number of bad and special k-neighbors adjacent to u, respectively. The superscript character G is often omitted if there is no ambiguity by the context. A triangle uvw is a [k, l, m]*-cycle* if $d_G(u) = k$, $d_G(v) = l$ and $d_G(w) = m$.

3. Proof of Theorem 1.3

For any graph G, set $n_i(G) = |\{v \mid \deg(v) = i\}|$ for $i = 1, 2, \ldots, \Delta(G)$. A graph G' is *smaller* than the graph G if any of the following is true.

- $|E(G')| < |E(G)|$;
- $|E(G)| = |E(G')|$ and $(n_t(G'), n_{t-1}(G'), \ldots, n_2(G'), n_1(G'))$ precedes $(n_t(G), n_{t-1}(G), \ldots, n_2(G), n_1(G))$ with respect to the lexicographic order, where $t = \max\{\Delta(G), \Delta(G')\}$.

A graph is *minimal* for a property when no smaller graph satisfies it.

We prove the Theorem 1.3 by contradiction. Let G be a minimal counterexample to Theorem 1.3, that is, the graph G does not admit a tnsd-k-coloring, while any other planar graph smaller than G admits a tnsd-k-coloring. In Section 3, we will frequently conduct some operations on G, such as deleting or contracting edges, to obtain a planar graph H smaller than G. Therefore, H admits a tnsd-k-coloring ϕ' by the minimality of G. If we can extend ϕ' to a tnsd-k-coloring ϕ for G, then by contradiction we will get some structural properties which are forbidden in G. Finally we apply the discharging method to obtain a contradiction to the planarity of graph G.

When we color an element $x \in V(G) \cup E(G) \backslash (V(H) \cup E(H))$, the colors of edges and vertices in H adjacent to or incident with x are *forbidden* for x. If we remove the forbidden colors from the color set, then the remaining colors are *available* colors for x. The principal tools used in this paper are originated from those in [1].