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Long rainbow paths and rainbow cycles in edge colored graphs – A survey

He Chen

School of Mathematics, Southeast University, Nanjing 210096, PR China

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ABSTRACT

Graph coloring problem and problem on the existence of paths and cycles have always been popular topics in graph theory. The problem on the existence of rainbow paths and rainbow cycles in edge colored graphs, as an integration of them, was well studied for a long period. In this survey, we will review known results on this subject. Because of the relationship between cycles and paths, we will review results on the existence of rainbow cycles (including rainbow Hamilton cycles, long rainbow cycles and rainbow cycles with given length) first, and then long rainbow paths (including rainbow Hamilton paths and other long rainbow paths).

1. Introduction

We use Bondy and Murty [8] for terminology and notations not defined here and consider simple graphs only, unless otherwise stated. Let $G = (V(G), E(G))$ be a graph. By an *edge coloring* of G we mean a function $C : E(G) \to \mathbb{N}$, the set of natural numbers. If G is assigned such a coloring, then we say that G is an *edge colored graph*. Denote the edge colored graph by (G, C), and call $C(e)$ the *color* of the edge $e \in E$. A subgraph of G is called monochromatic if all the edges of it have the same color. A subgraph of G is called *rainbow (heterochromatic, or multicolored)* if no two edges of it have the same color. For a vertex v of G, we say that color i is *presented at* vertex v if some edge incident with v has color i. The *color degree* $d^c(v)$ is the number of different colors that are presented at v, and the *color neighborhoodCN*(v) is the set of different colors that are presented at v. Given a positive integer k, an edge coloring C of graph G is said to be a *k-good coloring* if at least k different colors are presented at each vertex of G, i.e., $d^c(v) \geq k$ for any vertex $v \in V(G)$. An edge-coloring of G is *proper* if no two adjacent edges of G have the same color. If the edges of the complete graph K_n is colored so that no edge is colored more than $k = k(n)$ times, we refer to this as a *k-bounded coloring*. Given a positive integer n and a graph H, let $g(n, H)$ denote the maximum k such that there exists a k-good coloring of $E(K_n)$ containing no rainbow copy of H. In another word, any k-good coloring of $E(K_n)$ with $k > g(n, H)$ must contain a rainbow copy of H.

Graph coloring problem has always been a popular topic in graph theory. Many coloring problems, such as 4-color problem, chromatic number and edge chromatic number of graphs, etc., are all most well-known problems in graph theory. In 1930, the Ramsey number problem was proposed in [42] for the first time. From then on, lots of results on Ramsey number problem and related topics have been carried out. Therefore, monochromatic subgraphs which are related to Ramsey number and rainbow subgraphs which are related to anti-Ramsey number have been getting wide attentions. These problems were first studied in vertex colored graphs and then in edge colored graphs.

E-mail addresses: chenhe@seu.edu.cn, chenhe_ch@163.com

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Long paths and long cycles are another kind of classical problems in graph theory. In the middle of last century, many important theorems on long paths and long cycles were given. These theorems were generalized to heavy paths and heavy cycles in weighted graphs later, and lots of results were got. After that, it is natural to wonder whether these theorems could be generalized to long rainbow paths and cycles in colored graphs.

Besides cycles and paths, other kinds of subgraphs such as matchings and trees etc. were also well studied in edge colored graphs. See surveys [22,30].

In recent years, rainbow connection and rainbow vertex-connection are new hot topics in graph coloring. Therefore, lots of results on rainbow paths especially short rainbow paths were given (see surveys [34,37] and recent papers [35,36]). So in this survey we will consider long rainbow paths rather than all rainbow paths.

In this survey, we will focus on long rainbow paths and rainbow cycles in edge colored graphs. Since we can get a path easily by deleting an arbitrary edge from a cycle, there is a rainbow k-cycle means that there is a rainbow k-path. Therefore we will consider rainbow cycles first in Section 2, and then long rainbow paths in Section 3.

2. Rainbow cycles

Since the longest cycle a graph could have is a hamiltonian cycle, we will consider the rainbow Hamilton cycles first. The aim is to find some sufficient conditions for rainbow Hamilton cycles.

This problem was first mentioned by Erdös, Nestril and Rödl [16], they posed a question as follows.

Question 1 [16]**.** How large can integer k be, such that each k-bounded edge coloring of K_n contains a rainbow Hamilton cycle?

They proved that k can be any constant. Then Hahn and Thomassen [26] showed that k could grow as fast as $n^{1/3}$.

Theorem 2.1 *[26]***.** *There is a positive constant c such that if k and $n \geq ck^3$ are positive integers, then any k-bounded edge coloring of K_n contains a rainbow Hamilton cycle.*

Besides that, they also gave a conjecture.

Conjecture 1 *[26]***.** *The biggest k which guarantees each k-bounded edge coloring of K_n contains a rainbow Hamilton cycle is linear of n.*

From then on, many researchers worked on this problem. In [21], Frieze and Reed improved the growth rate of k.

Theorem 2.2 *[21]***.** *There is an positive constant A such that if n is sufficiently large and k is at most $\lceil \frac{n}{A \ln n} \rceil$, then any k-bounded edge coloring of K_n contains a rainbow Hamilton cycle.*

Later, in [4], Albert, Frieze and Reed made a big progress on this problem, they improved Theorem 2.2 and proved Conjecture 1.

Theorem 2.3 *[4]***.** *If n is sufficiently large and k is at most $\lceil cn \rceil$, where $c < 1/32$ then any k-bounded edge coloring of K_n contains a rainbow Hamilton cycle.*

In [4], the authors also considered the rainbow Hamilton cycle in edge-colored directed graphs.

Theorem 2.4 *[4]***.** *If n is sufficiently large and k is at most $\lceil cn \rceil$, where $c < 1/64$ then any k-bounded edge coloring of the edges of the complete digraph DK_n contains a rainbow Hamilton cycle.*

In [28], Jahanbekam and West studied a deeper problem, they studied the existence of t edge-disjoint rainbow Hamilton cycles.

Theorem 2.5 *[28]***.** *If $t \geq 1$ and $n \geq 2t+2$, then the maximum number of colors in an edge-coloring of K_n not having t edge-disjoint rainbow Hamilton cycles is no less than $\binom{n-1}{2} + t$. Moreover, if $n \geq 8t-1$, the maximum number of colors is $\binom{n-1}{2} + t$.*

This theorem implies that if K_n is edge colored by no less than $\binom{n-1}{2} + 2$ colors, then there must exist a rainbow Hamilton cycle.

All the results above considered the existence rainbow Hamilton cycles, while Frieze and Krivelevich [20] considered not only the existence of rainbow Hamilton cycle but also the existence of rainbow cycles of all sizes.

Theorem 2.6 *[20]***.** *There exists an absolute constant $c > 0$ such that if an edge coloring of K_n is cn-bounded, then there exist rainbow cycles of all sizes $3 \leq k \leq n$.*

Besides the rainbow Hamilton cycles, the existence of long rainbow cycles are also widely discussed.

In [17], Erdös, Simonovits and Sós made a conjecture on the existence of rainbow cycles of given length.

Conjecture 2 *[17]***.** *For $n \geq k \geq 3$, if K_n is edge colored with at least $n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) + O(1)$ colors, then there must exist a rainbow C_k.*